A necessary condition for the strong stability of finite difference scheme approximations for hyperbolic corner problems

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Abstract

This article study the strong stability of finite difference scheme approximations for hyperbolic systems of equations in the quarter space. The main result is that as in the continuous framework of PDE impose the so-called uniform GKS () condition (which is the condition characterizing the strong stability of the finite difference scheme approximations in the half space) on each side of the quarter space is not sufficient to ensure the strong stability of the scheme in the quarter space. We decribe in this paper a new necessary inversibility condition. This condition seems to be some discretized version of the condition imposed in [Osh73] in the PDE framework.

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Contents

1 Introduction

In this article we are interested in the strong stability of finite difference scheme approximations for hyperbolic systems of partial differential equations in the quarter space. Such systems of partial differential equations read:

$$\begin{cases} \partial_t u + A_1\partial_1 u + A_2\partial_2 u = f, & \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}_+^2,\\ B_1 u_{|x_1=0} = g_1, & \text{for } (t,x_2)\in\mathbb{R}_+\times\mathbb{R}_+,\\ B_2 u_{|x_2=0} = g_2, & \text{for } (t,x_1)\in\mathbb{R}_+\times\mathbb{R}_+,\\ u_{|t\leq 0} = 0, & \text{for } x\in\mathbb{R}_+^2, \end{cases} \tag{1}$$

where the matrices $A_1, A_2 \in \mathbf{M}_{N\times N}(\mathbb{R})$ and where the matrices enconding the boundary conditions B_1 and B_2 are respectively in $\mathbf{M}_{p_1\times N}(\mathbb{R})$ and $\mathbf{M}_{p_2\times N}(\mathbb{R})$. The integer p_1 (resp. p_2) equals the number of positive eigenvalues of A_1 (resp. A_2) and is the only number of boundary conditions ensuring that the system (1) is not overdetermined or underdetermined. Indeed, in the hyperbolic setting only the incoming components of the trace of the solution have to be specified by the boundary condition.

In all this article () we will restrict our attention to homogeneous initial conditions and to corner problems and associated finite difference schemes with only two space dimensions. The following analysis can be extended with minor changes to corner problems or schemes set in cylindrical domains, that is to say $\mathbb{R}_+^2\times\mathbb{R}^{d-2}$.

About results concerning the semi-group stability of finite difference schemes (that is stability when initial conditions are not zero) we refer to [CG11] and [Cou15] for results in the half space and to [Ben] to a recent extension of these theorems to finite difference schemes in the quarter space.

Our main motivation to study such schemes in the quarter space is that in numerical simulations of Cauchy problems, due to the impossibility to implement the full space $\mathbb{R}^d$, the space of simulation is restricted to a "big" box. Then there are two main possibilities, firstly we can stop the simulation as soon as the simulated solution hits the boundary or a corner of the box. This process leads to a maximal time of simulation which can virtually be smaller than the desired time of simulation. Secondly, we can impose some artificial boundary conditions, that are call absorbing or transparent boundary conditions, which tend to minimize the rellections of the computed solution against the artificial boundary (see for example [Ehr10]-[EM77]). As a consequence understand what are the admissible boundary and corner conditions leading to strong stability and to obtain a full characterization of these conditions is a first step in the study of absorbing/transparent boundary conditions for schemes set in a rectangle.() (ou clairement le virer)

Before to give a precise definition of strong stability for finite difference schemes associated to (1) is it interesting to give a brief overview about the analogous concept concerning partial differential equations. In this continous setting this concept is refered as strong well-posedness. This is the object of the next paragraph in which we will give some elements about strong well-posedness for corner and half space problems.

1.1 Strong well-posedness in corner and half space domains

In the author's knowledge the full characterization of the boundary conditions leading to a strongly well-posed system set in the quarter space has not been achieved yet. By strong well-posedness we mean that for all choice () of source terms f, g_1 and g_2 in L^2, the system (1) admits a unique solution u in L^2, with traces on the sides of the boundary of $\mathbb{R}_+^2$, which satisfies the following energy estimate: there exists $C>0$ such that for all $\gamma>0$ we have:

$$\begin{aligned}\gamma\|u\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+^2)}+\|u_{|x_1=0}\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)} \quad &+ \quad \|u_{|x_2=0}\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)}\\ &\leq \quad C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+^2)}+\|g_1\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)}+\|g_2\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)}\right),\end{aligned} \tag{2}$$

where for some Banach space X, $L^2_\gamma(X)$ denotes the L^2 space with weight $e^{-\gamma t}$.

In other words by strong well-posedness we mean existence and uniqueness of a solution which is controllable by the source terms of the problem. This definition of strong well-posedness for corner problems is a straightforward generalization of the concept of strong well-posedness for hyperbolic initial boundary value problems in the half space (see (4)) for which the weighted spaces L^2_γ appear naturally due to Laplace transform (see for example [BG07]-[CP81]).

1.1.1 Strong well-posedness of problems in the half space

The full characterization of the boundary conditions leading to strong well-posedness has been achieved in [Kre70] and it is known that strong well-posedness is equivalent to the so-called uniform Kreiss-Lopatinskii condition. This condition states that in the normal mode analysis no stable modes is solution of the homogeneous (on the boundary) problem in the half space. Roughly speaking it means that without source term on the boundary then the boundary can not, by itself, generates incoming non trivial () information.

More precisely if we consider the boundary value problem in the half space:

$$\begin{cases}\partial_t u+A_1\partial_1 u+A_2\partial_2 u=f, & \text{for } (t,x_1,x_2)\in\mathbb{R}\times\mathbb{R}_+\times\mathbb{R},\\ Bu_{|x_1=0}=g, & \text{for } (t,x_2)\in\mathbb{R}\times\mathbb{R},\\ u_{|t\leq 0}=0, & \text{for } (x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}.\end{cases} \tag{3}$$

We say that this boundary value problem is strongly well-posed if for all f, g in L^2_γ then there exists a unique solution of (3), u, satisfying the energy estimate: there exists $C>0$ such that for all $\gamma>0$ we have:

$$\gamma\|u\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})}+\|u_{|x_1=0}\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R})}\leq C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})}+\|g\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R})}\right). \tag{4}$$

Then by Laplace transform in time and Fourier transform in the tangential space variable x_2 this problem becomes:

$$\begin{cases}\frac{d}{dx_1}u=\mathscr{A}_1(\sigma,\xi_2)u+A_1^{-1}f, & \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R},\\ Bu_{|x_1=0}=g, & \text{for } (t,x_2)\in\mathbb{R}_+\times\mathbb{R},\end{cases} \tag{5}$$

where $\sigma:=\gamma+i\tau$, $\gamma>0$, $\tau\in\mathbb{R}$ is the dual variable of t and where $\xi_2\in\mathbb{R}$ is the dual variable of x_2. In (3), $\mathscr{A}_1(\sigma,\xi_2)$ is the so-called resolvent matrix defined by:

$$\mathscr{A}_1(\sigma,\xi_2):=-A_1^{-1}(\sigma+i\xi_2A_2).$$

When $\gamma > 0$, by hyperbolicity of (3), this matrix only has eigenvalues with nonzero real part (see [Her63]) and we can define the generalized stable (resp. unstable) eigenspaces $E^s(\sigma, \xi_2)$ (resp. $E^u(\sigma, \xi_2)$) as the eigenspace associated to the eigenvalues of positive (resp. negative) real part of $\mathscr{A}_1(\sigma, \xi_2)$. Then the analysis of [Kre70]-[Mét00] shows that these stable and unstable subspaces admit continuous extension up to $\gamma = 0$. The uniform Kreiss-Lopatinskii condition then states that:

$$\forall \gamma \geq 0, \tau \in \mathbb{R}, \xi_2 \in \mathbb{R},\ \ker B \cap E^s(\sigma, \xi_2) = \{0\}\,.$$

In particular, the uniform Kreiss-Lopatinskii condition states that the restriction of the boundary matrix B to the stable subspace $E^s(\sigma, \xi_2)$ is invertible.

1.1.2 Strong well-posedness of corner problems

About the characterization of strong well-posedness for corner problems the main contribution is due to Osher in [Osh73]-[Osh74]. In these articles the author imposes the uniform Kreiss-Lopatinskii condition on each side () of the boundary (this is easy to see, by finite speed of propagation arguments, that it is a necessary condition for the strong well-posedness of the corner problem). However the author also imposes a new inversibility condition determining the value of the trace of the solution on one side in terms of the source term on this side. The way to obtain this new (expected to be) necessary condition for strong well-posedness will be describe precisely in Section 2 so we will here only give the main ideas of the analysis.

The first step to obtain Osher corner condition is, as for problems in the half space, to perform a Laplace transform in time to replace the time derivative by a complex parameter. However, in the quarter space geometry the space variables x_1 and x_2 do not lie in $\mathbb{R}$ so we can not take Fourier transform to treat one of this variable as a parameter and study an ordinary differential equation reading under the form (5).

The way proposed in [Osh73] to overcome this difficulty is to extend the solution by zero for negative x_1. The extended solution then solves a boundary value problem in the half space where the source term in the interior depends on the value of the trace on $\{x_1 = 0\}$. We can then perform Fourier transform in the variable x_1, solve the obtained boundary value problem in the half space thanks to Duhamel formula and return in the x_1 variable by reverse Fourier transform. This process give us a compatibility condition between the value of the trace on $\{x_2 = 0\}$, in terms of the value of the trace on $\{x_1 = 0\}$. Then we reiterate these computations, but this time after an extension by zero for negative x_2, this gives a compatibility condition between the trace on $\{x_1 = 0\}$ and the trace on $\{x_2 = 0\}$. At last we combine these two conditions to show that the trace on $\{x_1 = 0\}$ has to satisfy an equation reading:

$$(I - \mathbb{T}_{2\to 1}\mathbb{T}_{1\to 2})u_{|x_1=0} = Pg_1, \tag{6}$$

where P is some Fourier multiplier and where $\mathbb{T}_{1\to 2}$ (resp. $\mathbb{T}_{2\to 1}$) is an operator taking in input the value of the trace on $\{x_1 = 0\}$ (resp. $\{x_2 = 0\}$) and gives in output the value of the trace on $\{x_1 = 0\}$ (resp. $\{x_1 = 0\}$). We refer to (20)-(22) for a precise expression of these operators.

Osher corner condition states that the operator $(I - \mathbb{T}_{2\to 1}\mathbb{T}_{1\to 2})$ is invertible on $L^2(\mathbb{R}_+)$. It is interesting to remark that as the uniform Kreiss-Lopatinskii condition, Osher's corner condition is an inversibility condition. However, this time we do not require the inversibility of a matrix but we require the inversibility of the Fourier integral operator $(I - \mathbb{T}_{2\to 1}\mathbb{T}_{1\to 2})$. That is why Osher's corner condition is much more technical than the uniform Kreiss-Lopatinskii condition and seems to be hard to check concretely.

Thanks to this condition Osher demonstrates a *apriori* energy estimate of the form (2) but which includes a non explicit number of loss of derivatives. By loss of derivative, we mean that to control the L^2-norm in the left hand side of (2) then we need to use higher order sobolev norms in the right hand side. Moreover in [Osh73]-[Osh74] nothing is say about the existence of a solution. So we are still far of a complete characterization of the boundary conditions leading to strong well-posedness. We refer to [[Ben15], Chapitre 4 and Chapitre 5] for a result showing the strong well-posedness of corner problems for a particular kind of boundary conditions (namely the strictly dissipative ones) and to some improvements of the result of [Osh73]. We also refer to [HR16] and to [HT14] for results of strong well-posedness for particular boundary conditions in trihedral corners and in a rectangle respectively.

1.2 Strong stability of finite difference schemes

In this paragraph we give the definition of strong stability that we will use for finite difference scheme approximations of (1) and we also describe our main result that states that, as in the continous framework, a new inversibility condition is necessary for a finite difference scheme approximation of (1) to be strongly stable.

But before that we give some results about the strong stability of finite difference scheme approximations in the half space.

1.2.1 Strong stability for schemes in the half space

We consider a cartesian discretization of the space $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ of the form of a collection of boxes $[n\Delta t, (n+1)\Delta t[\times [j_1\Delta x_1, (j_1+1)\Delta x_1[\times [j_2\Delta x_2, (j_2+1)\Delta x_2[$, where Δt, Δx_1 and Δx_2 are the steps of the discretization and where $n \in \mathbb{N}$, $j_2 \in \mathbb{Z}$ and j_1 is an integer (the precise set in which j_1 lie will be described in the following lines).

As usual in the study of difference scheme approximations for hyperbolic problems, it is natural to assume that these discretization parameters are linked by the so-called CFL (COURANT-FRIEDRICHS-LEWY) condition. That is to say that the ratios $\lambda_1 := \frac{\Delta t}{\Delta x_1}$ and $\lambda_2 := \frac{\Delta t}{\Delta x_2}$ are kept constant as the time step Δt goes to zero.

Let (U_j^n) (from now on $j := (j_1, j_2)$) be a sequence that approximates the value of the exact solution of the problem in the half space (3) on the cell $[n\Delta t, (n+1)\Delta t[\times [j_1\Delta x_1, (j_1+1)\Delta x_1[\times [j_2\Delta x_2, (j_2+1)\Delta x_2[$. More precisely, in the interior of the half space (indexed by $j_1 \geq 1$) the sequence (U_j^n) solves a finite difference scheme reading:

$$U_j^{n+s+1} + \sum_{\sigma=0}^{s} Q^\sigma U_j^{n+\sigma} = \Delta t f_j^{n+s+1}, \text{ for } n \geq 0, j_1 \geq 1 \text{ and } j_2 \in \mathbb{Z}, \tag{7}$$

where (f_j^n) is some discretization of the source term f and where the Q^σ are matrices that discretized the differential operator $A_1\partial_1 + A_2\partial_2$. In view of its definition, the scheme that we are considering is explicit with s time steps. The coefficients Q^σ are choosen under the form:

$$Q^\sigma := \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} A^{\sigma,\mu} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2},$$

where the $A^{\sigma,\mu}$ are given matrices in $\mathbf{M}_{N\times N}(\mathbb{R})$ and where $\mathbf{T}_1$ (resp. $\mathbf{T}_2$) is the unitary shift operator in the j_1(resp. j_2)-direction. Consequently, the finite scheme approximation (7) has stencil $(\ell_1 + r_1)$ (resp. $\ell_2 + r_2)$) in the j_1(resp. j_2)-direction.

One of the simplest examples of such discretization is the well-known Lax-Friedrichs scheme defined by:

$$\begin{aligned} U_j^{n+1} &= \frac{1}{4}\left[U_{j_1+1,j_2}^n + U_{j_1-1,j_2}^n + U_{j_1,j_2+1}^n + U_{j_1,j_2-1}^n\right] \\ &- \frac{\lambda_1}{2} A_1 \left[U_{j_1+1,j_2}^n - U_{j_1-1,j_2}^n\right] - \frac{\lambda_2}{2} A_2 \left[U_{j_1,j_2+1}^n - U_{j_1,j_2-1}^n\right], \end{aligned}$$

which can be express under the generic form (7) with $s = 0$, $\ell_1 = r_1 = \ell_2 = r_2 = 1$ and with suitable coefficients $A^{0,\mu}$.

An important remark is that if we assume that the U_j^p, $j_1 \geq 1$ are known up to the time $n+s$ then, in view of the definition of Q^σ, to compute the U_j^{n+s+1}, $j_1 \geq 1$ it is necessary to know the U_j^n for $1-\ell_1 \leq j_1 \leq 0$. That is why we have to impose some extra equations determining these terms. This is done by choosing a discretization of the boundary condition on $\{x_1 = 0\}$. In the litterature, the considered discretizations of the boundary condition read:

$$U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B^{\sigma,j_1} U_{1,j_2}^{n+\sigma} = g_j^{n+s+1}, \text{ for } n \geq 0, 1-\ell_1 \leq j_1 \leq 0 \text{ and } j_2 \in \mathbb{Z}, \tag{8}$$

where the matrices $B^{\sigma,j_1} \in \mathbf{M}_{N\times N}(\mathbb{R})$ are given by:

$$B^{\sigma,j_1} := \sum_{\mu_1=0}^{b_1} \sum_{\mu_2=-b_2}^{b_2} B^{\sigma,\mu,j_1} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2},$$

and where g_j^{n+s+1} stands for some discretization of the source term g.

This choice of discretization of the boundary condition permits to compute the U_j^{n+s} for $1-\ell_1 \leq j_1 \leq 0$ and consequently the scheme (7)-(8) effectively permits to determine U_j^{n+s+1}.

We remark that contrary to the system of partial differential equations (3), the boundary condition (8) involves all the components of the trace on $\{x_1 = 0\}$. This is a substantial difference between initial boundary value problems and finite difference scheme approximations.

However, as in the continous setting, the full characterization of the discretized boundary conditions (8) leading to strong stability has been established in [BGS72] and then generalized in [Cou09]-[Cou11] (at least in $\mathbb{R}_+ \times \mathbb{R}$). The condition ensuring the strong stability of (7)-(8) is the so-called discrete uniform Kreiss-Lopatinskii (or uniform Godunov-Ryabenkii) condition.

By strong stability for the scheme (7)-(8) (with homogeneous initial conditions) we mean that the solution (U_j^n) satisfies some discretized version of the energy estimate (4). More precisely we ask that there exists $C > 0$ such that for all $\gamma > 0$ and all $\Delta t \in [0, 1[$:

$$\begin{aligned}
&\frac{\gamma}{\gamma\Delta t + 1} \sum_{n\geq s+1} \sum_{j_1 \geq 1-\ell_1} \sum_{j_2\in\mathbb{Z}} \Delta t \Delta x e^{-2\gamma n\Delta t} |U_j^n|^2 + \sum_{n\geq s+1} \sum_{j_1=1-\ell_1}^{r_1} \sum_{j_2\in\mathbb{Z}} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} |U_j^n|^2 \\
&\quad \leq C \left(\frac{\gamma\Delta t + 1}{\gamma} \sum_{n\geq s+1} \sum_{j_1\geq 1} \sum_{j_2\in\mathbb{Z}} \Delta t \Delta x e^{-2\gamma n\Delta t} |f_j^n|^2 + \sum_{n\geq s+1} \sum_{j_1=1-\ell_1}^{0} \sum_{j_2\in\mathbb{Z}} \Delta t \Delta x_2 e^{-2\gamma n\Delta t} |g_j^n|^2 \right),
\end{aligned}$$

where $\Delta x := \Delta x_1 \Delta x_2$.

The way to obtain the uniform Kreiss-Lopatinskii condition for (7)-(8) is more or less the same than the one to derive the uniform Kreiss-Lopatinskii condition in the continous setting. Firstly we replace the "discrete derivative" in time by a complex parameter z, and to replace the "discrete derivative" in x_2 by a real parameter, ξ_2. Then this new scheme is expressed in terms of an augmented vector $\mathscr{X}_{j_1}$:

$$\begin{cases} \mathscr{X}_{j_1+1} = \mathbb{M}(z,\xi_2)\mathscr{X}_{j_1} + \mathcal{F}_{j_1}\ , \ \text{ for } j_1 \geq 1, \\ \mathbb{B}(z,\xi_2)\mathscr{X}_1 = \mathcal{G}, \end{cases} \tag{9}$$

for suitable matrices $\mathbb{M}$, $\mathbb{B}$ and source terms $(\mathcal{F}_{j_1})$ and $\mathcal{G}$. We refer to Section 6.1 or [[Cou13], Paragraph 2.3.3] for more details about this construction.

Then we can show that, for z with modulus greater than one and for ξ_2 in $\mathbb{R}$, the matrix $\mathbb{M}(z,\xi_2)$ only admits stable (that is to say with modulus less than one) or unstable (with modulus greater than one) eigenvalues. The eigenspace associated to the stable eigenvalues is denoted by $\mathbb{E}^s(z,\xi_2)$. It can also be show that the stable subspace $\mathbb{E}^s(z,\xi_2)$ admits continous extensions up to z with modulus one.

Thus the uniform Kreiss-Lopatinskii for finite difference schemes states that:

$$\forall z \in \mathbb{C}, |z| \geq 1, \forall \xi_2 \in \mathbb{R},\ \mathbb{E}^s(z,\xi_2) \cap \ker \mathbb{B}(z,\xi_2) = \{0\}\,,$$

and as to be compared with the uniform Kreiss-Lopatinskii condition in the continous setting.

1.2.2 Strong stability for schemes in the quarter space

The litterature about the question of the strong stability for schemes in the quarter space is much more poor than in the half space and in the author's knowledge this question is widely open.

In this article we will see how the work of [Osh73] can be translated in the discrete setting which will permit us to exhibated a new necessary condition for strong well-posed of finite difference schemes in the quarter space. However the question of the sufficiency of this condition is left to future works.

Schemes that we will consider read under the form:

$$\begin{cases} U_j^{n+s+1} + \sum_{\sigma=0}^{s} Q^\sigma U_j^{n+\sigma} = \Delta t f_j^{n+s+1}, & \text{for } j_1, j_2 \geq 1, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} U_{1,j_2}^{n+\sigma} = g_{1,j}^{n+s+1}, & \text{for } 1-\ell_1 \leq j_1 \leq 0, j_2 \geq 1, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_2^{\sigma,j_2} U_{j_1,1}^{n+\sigma} = g_{2,j}^{n+s+1}, & \text{for } j_1 \geq 1, 1-\ell_2 \leq j_2 \leq 1, n \geq 0, \\ U_j^n = 0, & \text{for } j \in \mathscr{R}, n \in [\![0, s]\!], \end{cases}$$

and thus are the most natural generalization of the scheme (7)-(8) from the half space to the quarter space geometry. However and extra condition determining the discretization of the solution near the corner will also have to be specified (see Paragraph 3.2 for more details).

The way to derive the new inversibility condition will follow the analysis in the continous setting. That is to say that we will firstly replace the " discrete time derivative" by a complex valued paramater. Then we will extend the solution by zero for $j_1 \leq 1-\ell_1$ in order to obtain a scheme set in the half space $j_2 \geq 1-\ell_2$.

This extension will, as in the continous setting, induce an error term in the interior (that is for $j_2 \geq 1$) that can be seen as the discretized value of the trace of (U_j^n) on the side " $\{x_1 = 0\}$ ". However do to the discrete nature of the scheme this error will involve several values of the (U_j^n) and not only the values for $j_1 = 0$. It will also have a more delicate expression than in the continous setting.

An other notable fact is that, compared to the continous setting, this extension will also induce an error term on the boundary (that is for $1-\ell_2 \leq j_2 \leq 0$). However, this extra error term on the boundary will also depends on the value of the trace of (U_j^n) on the side " $\{x_1 = 0\}$ ".

Once we are in the half space geometry, the variable j_1 will be replaced by a real parameter ξ_1 by Fourier transform and this we will have to study a scheme reading under the form (9). Such a problem will be easily solvable by discrete Duhamel's formula and this will permits to express the value of the trace of (U_j^n) on the side " $\{x_2 = 0\}$ " in terms of the errors in the interior and on the boundary and thus in terms of the trace of (U_j^n) on the side " $\{x_1 = 0\}$ ".

We will then reiterate the process but by extending the solution by zero for $j_2 \leq 1-\ell_2$ this time to derive a compatibility condition between the trace of (U_j^n) on " $\{x_1 = 0\}$ " in terms of the trace of (U_j^n) on the side " $\{x_2 = 0\}$ ".Combining these two relations gives a compatbility condition for the trace of (U_j^n) on the side " $\{x_1 = 0\}$ " which will be similar to the one obtain in the continous setting (see (6)) except for some new terms induced by the erros on the sides.

1.3 Organization of the article and notations

1.3.1 Organization of the paper

The paper is organized as follows, in Section 2 we describe the way to derive the so-called Osher's corner condition for strong well-posedness for PDE. This section gives the main steps of the construction that we will adapt to finite difference schemes.

Then in Section 3 we describe the schemes that we will consider and state the main assumptions.

Section 4 is devoted to the first reduction of the finite difference scheme, that is to say that the strong stability for the considered scheme is equivalent to the strong stability of a so-called time resolvent scheme (for which the time variable n is replace) by a complex parameter.

Section 5 aims to extend the time resolvent scheme to a scheme in the half space by considering the extension of the solution by zero for negative values of j_1. This procedure is the main step to derive Osher's corner condition for PDE. However, in the setting of finite difference scheme, as the reader will see, the extension is much more delicated and will lead us to more complicated error source terms in the interior and on the boundary.

Then Section 6 describes the new necessary inversibility condition for strong stability.

At last, Section 7 contains some explicit computations and numerical results.

1.3.2 Notations

In all what follows we define the following subsets of $\mathbb{C}$:

$$\mathscr{U} := \{z \in \mathbb{C} |\, |z| > 1\}\,, \mathbb{D} := \{z \in \mathbb{C} |\, |z| < 1\}\,.$$

For $a, b \in \mathbb{R}$, we will use the short hand notation $[\![a, b]\!]$ for the "intervals of integers". More precisely, $[\![a, b]\!] := \mathbb{Z} \cap [a, b]$.

2 Osher's corner condition for PDE

This section describes how the corner condition for hyperbolic corner problems in the continous framework is obtained in [Osh73]. As we have already mentionned, the derivation of the necessary condition for strong stability for finite difference schemes will follows the same lines. Consequently this section is a good introduction before we turn to finite difference schemes.

We consider the hyperbolic corner problem:

$$\begin{cases} (\partial_t + A_1\partial_1 + A_2\partial_2)u = f, & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}_+^2, \\ B_1 u_{|x_1=0} = g_1, & \text{for } (t, x_2) \in \,]-\infty, T] \times \mathbb{R}_+, \\ B_2 u_{|x_2=0} = g_2, & \text{for } (t, x_1) \in \,]-\infty, T] \times \mathbb{R}_+, \\ u_{|t\leq 0} = 0, & \text{for } x \in \mathbb{R}_+^2, \end{cases} \tag{10}$$

for which we will make the following assumptions:

Assumption 2.1 (Constant hyperbolicity) *There exists $M \in \mathbb{N}^*$, analytic functions on $\mathbb{R}^d \backslash \{0\}$ denoted by $\lambda_1, ..., \lambda_M$ with real values and positive integers $\nu_1, ..., \nu_M$ such that:*

$$\forall \xi \in \mathbb{S}^1, det\,(\tau + \xi_1 A_1 + \xi_2 A_2) = \prod_{k=1}^{M} (\tau - \lambda_k(\xi))^{\nu_k},$$

where $\lambda_1(\xi) < ... < \lambda_M(\xi)$ and such that the eigenvalues $\lambda_k(\xi)$ of $\sum_{j=1}^{2} \xi_j A_j$ are semi-simple.

and

Assumption 2.2 (Non characteristic boundaries) *The matrices A_1 and A_2 are invertible.*

We also introduce the following definition of strong well-posedness for corner problems which specifies the one given in the introduction:

Definition 2.1 (Strong well-posedness) *We say that the corner problem* (10) *is strongly well-posed if for all source terms $f \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+^2)$, $g_1 \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$, $g_2 \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$, the system* (10) *admits a unique solution $u \in L^2_\gamma(\mathbb{R}_+^2)$, with traces in $L^2_\gamma(\mathbb{R}_+)$ satisfying the energy estimate: there exists $C > 0$ such that for all $\gamma > 0$,*

$$\begin{aligned} \gamma \|u\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+^2)} \quad &+ \quad \|u_{|x_1=0}\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)} + \|u_{|x_2=0}\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)} \\ &\leq \quad C\left(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+^2)} + \|g_1\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)} + \|g_2\|^2_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)}\right), \end{aligned} \tag{11}$$

where the weighted L^2 spaces, $L^2_\gamma(X)$ (here X denotes a Banach space), are defined by:

$$L^2_\gamma(X) := \left\{u \in \mathscr{D}(X) | e^{-\gamma t} u \in L^2(X)\right\},$$

with norm:

$$\|\cdot\|_{L^2_\gamma(X)} := \|e^{-\gamma t}\cdot\|^2_{L^2(X)}.$$

The first step to obtain the corner condition is to "kill" the time variable by Laplace transform. This point is made in the following paragraph.

2.1 The time-resolvent PDE

We denote by $\widehat{\cdot}$ the Laplace transform in the time variable and $v := \widehat{u}$ where u is the solution of (10). Let $\sigma := \gamma + i\tau$, with $\gamma > 0$ be the dual variable of t. Then v solves the boundary value problem (remark that we have homogeneous initial conditions):

$$\begin{cases} \sigma v + A_1\partial_1 v + A_2\partial_2 v = \widehat{f}, & \text{for } x \in \mathbb{R}^2_+, \\ B_1 v_{|x_1=0} = \widehat{g_1}, & \text{for } x_2 \in \mathbb{R}_+, \\ B_2 v_{|x_2=0} = \widehat{g_2}, & \text{for } x_1 \in \mathbb{R}_+. \end{cases} \tag{12}$$

The boundary value problem (12) is refered as the time-resolvent corner problem.

We introduce the following concept of strong stability for the time-resolvent corner problem:

Definition 2.2 *We say that the time resolvent corner problem* (12) *is strongly well-posed if for all source terms $f \in L^2(\mathbb{R}^2_+)$, $g_1, g_2 \in L^2(\mathbb{R}_+)$, the system* (12) *admits a unique solution $v \in L^2(\mathbb{R}^2_+)$, with traces in $L^2(\mathbb{R}_+)$ satisfying the energy estimate: there exists $C > 0$ such that for all $\gamma > 0$,*

$$\gamma\|v\|^2_{L^2(\mathbb{R}^2_+)} + \|v_{|x_1=0}\|^2_{L^2(\mathbb{R}_+)} + \|v_{|x_2=0}\|^2_{L^2(\mathbb{R}_+)} \leq C\left(\frac{1}{\gamma}\|\widehat{f}\|^2_{L^2(\mathbb{R}^2_+)} + \|\widehat{g_1}\|^2_{L^2(\mathbb{R}_+)} + \|\widehat{g_2}\|^2_{L^2(\mathbb{R}_+)}\right). \tag{13}$$

The main result of this paragraph is the following and states that the study of the strong stability (10) of reduces to the study of the strong stability of (12). More precisely:

Proposition 2.1 ([Ben15], Proposition 5.3.1) *The corner problem* (10) *is strongly well-posed in the sense of Definition 2.1 if and only if the time-resolvent corner problem* (12) *is strongly well-posed in the sense of Definition 2.2.*

Proof : The fact that the strong well-posedness of (12) implies the well-posedness of (10) is a direct consequence of Paley-Wiener theorem.

We turn to the proof of "(10) well posed implies (12) well-posed". The Laplace transform in time of u the solution of (10) solves (12) so a solution exists it remains to show the energy estimate (13). Indeed the uniqueness follows from the linearity and (13).

Let $\underline{\sigma} := \underline{\gamma} + i\underline{\tau}$, with $\underline{\gamma} > 0$ and $\underline{\tau} \in \mathbb{R}$ be a fixed Laplace parameter. We introduce a sequence of functions $(\varphi_n)_{n\in\mathbb{N}} \in \mathscr{C}^\infty_c(\mathbb{R})$ satisfying that there exists $C_0 > 0$ independent of n such that:

$$\begin{cases} \varphi_n(t) = 1, & \text{for } |t| \leq n, \\ \varphi_n(t) = 0, & \text{for } |t| \geq n+1, \\ \varphi_n(t) \in [0,1], & \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}, \\ |\varphi_n'(t)| \leq C_0, & \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}. \end{cases}$$

We define a sequence of functions $(v_n)_{n\in\mathbb{N}}$ by:

$$v_n(t,x) := \frac{e^{\underline{\sigma} t}}{\sqrt{2n}}\varphi_n(t)v(x),$$

where we recall that v denotes the Laplace transform in time of u. We have:

$$L(\partial)v_n = \frac{e^{\sigma t}}{\sqrt{2n}}\left(\varphi_n \widehat{f} + \varphi_n' v\right) := f_n,$$

and for $k \in \{1,2\}$:

$$B_k v_{n_{|x_k=0}} = \frac{e^{\sigma t}}{\sqrt{2n}}\varphi_n \widehat{g_k} := g_{k,n}.$$

By assumption the corner problem (10) is strongly well-posed so thanks to the energy estimate (11) we have:

$$\begin{aligned} \underline{\gamma}\|v_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} &+ \|v_{n_{|x_1=0}}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}_+)} + \|v_{n_{|x_2=0}}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}_+)} \\ &\leq C \left(\frac{1}{\underline{\gamma}}\|f_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} + \|g_{1,n}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}_+)} + \|g_{2,n}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}_+)}\right). \end{aligned} \tag{14}$$

A direct computation gives:

$$\|v_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \frac{1}{2n}\left(\int_{\mathbb{R}} |\varphi_n(t)|^2\, dt\right)\|v\|^2_{L^2(\mathbb{R}^2_+)},$$

and by definition of $(\varphi_n)_{n\in\mathbb{N}}$ we have:

$$2n \leq \int_{\mathbb{R}} |\varphi_n(t)|^2\, dt \leq 2(n+1). \tag{15}$$

So we immediately deduce that

$$\lim_{n\to+\infty}\|v_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \|v\|^2_{L^2(\mathbb{R}^2_+)}.$$

The same kind of computations also gives, that for $k \in \{1,2\}$,

$$\lim_{n\to+\infty}\|v_{n_{|x_k=0}}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \|v_{|x_k=0}\|^2_{L^2(\mathbb{R}^2_+)} \text{ and } \lim_{n\to+\infty}\|g_{k,n}\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \|\widehat{g_k}\|^2_{L^2(\mathbb{R}^2_+)}.$$

Consequently to establish (13) it is sufficient to show that $\lim_{n\to+\infty}\|f_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \|\widehat{f}\|^2_{L^2(\mathbb{R}^2_+)}$, and to take the limit $n\uparrow\infty$ in (14).

We decompose

$$\|f_n\|^2_{L^2_{\underline{\gamma}}(\mathbb{R}\times\mathbb{R}^2_+)} = \|f_n\|^2_{L^2_{\underline{\gamma}}([-n,n]\times\mathbb{R}^2_+)} + \|f_n\|^2_{L^2_{\underline{\gamma}}(C_n\times\mathbb{R}^2_+)},$$

where $C_n := [-(n+1), n+1]\setminus[-n,n]$. Using the fact that $\varphi' \equiv 0$ on $[-n,n]$ and the bounds (15) we obtain that

$$\lim_{n\to+\infty}\|f_n\|^2_{L^2_{\underline{\gamma}}([-n,n]\times\mathbb{R}^2_+)} = \|\widehat{f}\|^2_{L^2(\mathbb{R}^2_+)},$$

so it remains to show that $\lim_{n\to+\infty}\|f_n\|^2_{L^2_{\underline{\gamma}}(C_n\times\mathbb{R}^2_+)} = 0$. In view of the definition of $(f_n)_{n\in\mathbb{N}}$ and the properties imposed on $(\varphi_n)_{n\in\mathbb{N}}$ we have:

$$\|f_n\|^2_{L^2_{\underline{\gamma}}(C_n\times\mathbb{R}^2_+)} \leq \frac{MK}{n},$$

where $M := \max(\|v\|^2_{L^2(\mathbb{R}^2_+)}, \|\widehat{f}\|^2_{L^2(\mathbb{R}^2_+)})$ and $K := \max(C_0, 1)$. It follows that $\lim_{n\to+\infty}\|f_n\|^2_{L^2_{\underline{\gamma}}(C_n\times\mathbb{R}^2_+)} = 0$ and this concludes the proof of (13).

$\square$

2.2 The corner condition of [Osh73]

Once we have restricted the study to question of the well-posedness of the time-resolvent corner problem, we can describe the new inversibility imposed in [Osh73]. To do this we consider the following time-resolvent corner problem:

$$\begin{cases} \sigma v + A_1\partial_1 v + A_2\partial_2 v = 0, & \text{for } x \in \mathbb{R}_+^2, \\ B_1 v_{|x_1=0} = g_1, & \text{for } x_2 \in \mathbb{R}_+, \\ B_2 v_{|x_2=0} = 0, & \text{for } x_1 \in \mathbb{R}_+, \end{cases} \tag{16}$$

where the Laplace parameter σ now acts like a parameter.

We assume that (16) is strongly well-posed in the sense of Definition (2.2) and we define w the extension of v the solution of (16) by zero for negative x_1. This extension solves the following boundary value problem in the half space $\{x_1 \in \mathbb{R}, x_2 \geq 0\}$:

$$\begin{cases} \sigma w + A_1\partial_1 w + A_2\partial_2 w = \delta_0(x_1)A_1 v_{|x_1=0}, & \text{for } x_1 \in \mathbb{R}, x_2 \geq 0, \\ B_2 v_{|x_2=0} = 0, & \text{for } x_1 \in \mathbb{R}. \end{cases} \tag{17}$$

The variable x_1 now lie in the full space $\mathbb{R}$ so we can perform the Fourier transform of w with respect to x_1. Indeed by strong well posednes $v \in L^2(\mathbb{R}_+^2)$ and consequently $w \in L^2(\mathbb{R}\times\mathbb{R}_+)$. Let ξ_1 be the dual variable of x_1 and $\widehat{\cdot}$ denotes the x_1-Fourier transform. Then $\widehat{w}$ is solution of:

$$\begin{cases} \partial_2 \widehat{w} = \mathscr{A}_2(\sigma,\xi_1)\widehat{w} + A_2^{-1}A_1 v_{|x_1=0}, & \text{for } x_1 \in \mathbb{R}, x_2 \geq 0, \\ B_2 \widehat{w}_{|x_2=0} = 0, & \text{for } x_1 \in \mathbb{R}, \end{cases} \tag{18}$$

where $\mathscr{A}_2(\sigma,\xi_1)$ is the so-called resolvent matrix defined by:

$$\mathscr{A}_2(\sigma,\xi_1) := -A_2^{-1}\left(\sigma + i\xi_1 A_1\right).$$

Remark in particular that to derive (18) we used the non-charcteristicity assumption 2.2. This assumption will also have its discrete version for finite diffrence schemes (see () for more details).

Before to give a solution of (18) thanks to Duhamel's formula we have to study the eigenvalues/eigenspaces of the resolvent matrix $\mathscr{A}_2(\sigma,\xi_1)$. The following result due to [Her63] is classical and admits an analogous version for finite difference schemes (see Lemma 6.1).

Lemma 2.1 ([Her63]) *Assume that A_2 is invertible and that the corner problem (10) satisfies Assumption 2.1. Then for all $\gamma > 0, \xi_1 \in \mathbb{R}, \tau \in \mathbb{R}$, the eigenvalues of $\mathscr{A}_2(\sigma,\xi_1)$ have non zero real parts. Moreover the number of eigenvalues with strictly negative real part is constant and is equals to p_2.*

As a consequence we have the following decomposition:

$$\forall \sigma \in \mathbb{C} \text{ such that } \gamma > 0,\ \forall \xi_1 \in \mathbb{R},\ \mathbb{C} = E_2^s(\sigma,\xi_1) \oplus E_2^u(\sigma,\xi_1),$$

where $E_2^s(\sigma,\xi_1)$ (resp. $E_2^u(\sigma,\xi_1)$) denotes the stable (resp. unstable) subspace of $\mathscr{A}_2(\sigma,\xi_1)$ that is the generalized eigenspace associated to eigenvalues with negative (resp. positive) real parts.

The proof is classical but we will give because it is a good introduction for the proof of the analogous version of Lemma 2.1 in the discrete setting (see Lemma 6.1).

Proof : Let $\lambda \in i\mathbb{R}$ be an eigenvalue of $\mathscr{A}_2(\sigma,\xi_1)$ for some $\gamma > 0$, then we have:

$$0 = \det(\mathscr{A}_2(\sigma,\xi_1) - \lambda I) = (-1)^d \det A_2^{-1} \det(\sigma + i\xi_1 A_1 + \lambda A_2).$$

Consequently σ is an eigenvalue of $\xi_1 A_1 + \frac{\lambda}{i}A_2$. By Assumption 2.1, it implies that $\sigma \in i\mathbb{R}$ which is incompatible with the fact that $\gamma > 0$.

We now justify that the number of eigenvalues with strictly negative real part is equals to p_2. The application that $(\sigma,\xi_1)\mapsto \det(\mathscr{A}_2(\sigma,\xi_1)-\lambda I)$ is continous with constant degree. So the number of roots λ (counted with multiplicity) with negative real part may not vary locally. The set of parameters $\{z\in\mathbb{C}, \mathrm{Re}\ z>0\}\times\mathbb{R}$ is connected so the number of roots with negative real part must be constant on $\{z\in\mathbb{C}, \mathrm{Re}\ z>0\}\times\mathbb{R}$. Evaluating in $(1,0)$, we obtain:

$$0=\det(\mathscr{A}_2(1,0)-\lambda I)=(-1)^d\det(A_2^{-1}+\lambda I),$$

which admits p_2 strictly negative real roots.

□

We introduce $\Pi_2^s(\sigma,\xi_1)$ (resp. $\Pi_2^u(\sigma,\xi_1)$) the projector upon $E_2^s(\sigma,\xi_1)$ (resp. $E_2^u(\sigma,\xi_1)$) according to the decomposition given in Lemma 2.1.

We can decompose $\widehat{w}$, the solution of (18) into a stable and an unstable part:

$$\widehat{w}(\xi_1,x_2):=\Pi_2^s(\sigma,\xi_1)\widehat{w}(\xi_1,x_2)+\Pi_2^u(\sigma,\xi_1)\widehat{w}(\xi_1,x_2),$$

where each part is given by the Duhamel's formula:

$$\begin{aligned}\Pi_2^s(\sigma,\xi_1)\widehat{w}(\xi_1,x_2) &= e^{x_2\mathscr{A}_2(\sigma,\xi_1)}\Pi_2^s(\sigma,\xi_1)\widehat{w}(\xi_1,0)+\int_0^{x_2}e^{(x_2-s)\mathscr{A}_2(\sigma,\xi_1)}\Pi_2^s A_2^{-1}A_1 v_{|x_1=0}(s)\,ds,\\ \Pi_2^u(\sigma,\xi_1)\widehat{w}(\xi_1,x_2) &= -\int_{x_2}^{+\infty}e^{(x_2-s)\mathscr{A}_2(\sigma,\xi_1)}\Pi_2^s A_2^{-1}A_1 v_{|x_1=0}(s)\,ds.\end{aligned}$$

But from (), we know that from the Kreiss-Lopatinskii condition, at the level of the boundary $\{x_2=0\}$, the stable part of the trace depends on the unstable part. More precisely we can write:

$$\Pi_2^s(\sigma,\xi_1)\widehat{w}(\xi_1,0)=-\phi_2(\sigma,\xi_1)B_2\Pi_2^u(\sigma,\xi_1)\widehat{w}(\xi_1,0),$$

where we recall that $\phi_2(\sigma,\xi_1)=B^{-1}_{2|E_2^s(\sigma,\xi_1)}$. So the expression of the trace of $\widehat{w}$ on the boundary $\{x_2=0\}$ becomes:

$$\begin{aligned}\widehat{w}(\xi_1,0) &= [I-\phi_2(\sigma,\xi_1)B_2]\,\Pi_2^u(\sigma,\xi_1)\widehat{w}(\xi_1,0),\\ &= [\phi_2(\sigma,\xi_1)B_2-I]\int_0^{+\infty}e^{-s\mathscr{A}_2(\sigma,\xi_1)}\Pi_2^u A_2^{-1}A_1 v_{|x_1=0}(s)\,ds.\end{aligned}$$

By reverse Fourier transform of the previous expression it follows that:

$$w_{|x_2=0}(x_1)=(T_{1\to 2}(\sigma)v_{|x_1=0})(x_1), \tag{19}$$

where the operator $T_{1\to 2}(\sigma)$ is a Fourier integral operator defined by:

$$(T_{1\to 2}(\sigma)u):=\frac{1}{2\pi}\int_{\mathbb{R}}e^{ix_1\xi_1}\,[\phi_2(\sigma,\xi_1)B_2-I]\int_0^{+\infty}e^{-s\mathscr{A}_2(\sigma,\xi_1)}\Pi_2^u A_2^{-1}A_1 u(s)\,ds\,d\xi_1. \tag{20}$$

However by definition of w we have $w_{|x_2=0}=v_{|x_2=0}$ so (19) is a compatibility relation between $v_{|x_2=0}$ and $v_{|x_1=0}$. More precisely it says that the value of the trace of the solution on the side $\{x_2=0\}$ is given by the value of the trace on the side $\{x_1=0\}$ under the action of the operator $T_{1\to 2}(\sigma)$.

Then we can reiterate exactly the same construction but after an extension by zero for x_2 negative to obtain an analogous relation to (19) which associates the value of the trace on $\{x_1=0\}$ in terms of the value of the trace on $\{x_2=0\}$:

$$v_{|x_1=0}(x_2)=(T_{2\to 1}(\sigma)v_{|x_2=0})(x_2)+(P(\sigma)g_1)(x_2), \tag{21}$$

where the operator $T_{2\to1}(\sigma)$ is a Fourier integral operator whose expression is similar to the expression of $T_{1\to2}(\sigma)$:

$$(T_{2\to1}(\sigma)u)(x_2) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}\left[\phi_1(\sigma,\xi_1)B_1 - I\right]\int_0^{+\infty} e^{-s\mathscr{A}_1(\sigma,\xi_2)}\Pi_1^u(\sigma,\xi_2)A_1^{-1}A_2u(s)\,ds\,d\xi_2, \tag{22}$$

with $\mathscr{A}_1(\sigma,\xi_2) := -A_1^{-1}\left(\sigma + i\xi_2A_2\right)$, and where $\Pi_1^u(\sigma,\xi_2)$ is the projector upon $E_1^u(\sigma,\xi_2)$ the unstable subspace of $\mathscr{A}_1(\sigma,\xi_2)$ (see Lemma 2.1).

In (21), $P(\sigma)$ is a Fourier multiplier defined by:

$$(P(\sigma)g)(x_2) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}\phi_1(\sigma,\xi_2)g(\xi_2)\,d\xi_2.$$

Combining (19) and (21) leads to the following necessary compatibility condition on the trace $v_{|x_1=0}$:

$$\left[I - \mathbb{T}_1(\sigma)\right]v_{|x_1=0} = P(\sigma)g_1, \tag{23}$$

where we defined $\mathbb{T}_1(\sigma) := T_{2\to1}(\sigma)T_{1\to2}(\sigma)$.

If one believes that the value of the source term g_1 on the side $\{x_1 = 0\}$ determines the value of the trace $v_{|x_1=0}$ it is thus natural to ask that the operator $[I - \mathbb{T}_1(\sigma)]$ is invertible on $L^2(\mathbb{R}_+)$ with values in $L^2(\mathbb{R}_+)$. Moreover as in the energy estimate defining the well-posedness of the time-resolvent system (13) we ask that the constant C does not depends on γ. Finally we ask that $[I - \mathbb{T}_1(\sigma)]$ is uniformly invertible compared to σ.

It is the so-called Osher's corner condition:

Definition 2.3 ([Osh73]) *We say that the corner problem* (10) *satisfies the Osher's corner condition if for all $\sigma := \gamma + i\tau$ with $\gamma > 0$, the operator $[I - \mathbb{T}_1(\sigma)]$ is invertible on $L^2(\mathbb{R}_+)$ uniformly in terms of σ.*

In particular there exists a constant $C > 0$ such that for all σ with $\gamma > 0$ and for all $u \in L^2(\mathbb{R}_+)$ we have:

$$\|u\|^2_{L^2(\mathbb{R}_+)} \leq C\|\left[I - \mathbb{T}_1(\sigma)\right]u\|^2_{L^2(\mathbb{R}_+)}.$$

Under this condition, it is possible to construct a "Kreiss type symmetrizor" to show an *a priori* energy estimate for the solution u, see [Osh73]. However, the energy estimate obtained in [Osh73] contains a non explicit number of losses of derivatives and consequently is not suitable to characterize the strong well-posedness.

So note that neither the necessary nature nor of the sufficient nature of Osher's corner condition has been rigorously demonstrate yet. In the author's opinion it is due to two main difficulties. On the one hand we do not, at the present time, understand sufficiently well their influence upon the traces on the solution. And one the other hand, generically the operators $T_{2\to1}(\sigma)$ and $T_{2\to1}(\sigma)$ are not explicitely computable.

Indeed, in the author's knowledge, the only example where the operators $T_{2\to1}(\sigma)$ and $T_{2\to1}(\sigma)$ are computable is the quite simple system:

$$\begin{cases} \partial_t u + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\partial_1 u + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\partial_2 u = 0, \\ \begin{bmatrix} 1 & -a \end{bmatrix} u_{|x_1=0} = g_1, \\ \begin{bmatrix} -b & 1 \end{bmatrix} u_{|x_2=0} = 0, \\ u_{|t\leq 0} = 0, \end{cases}$$

for which it can be shown (see [**?**]) that the strong well posedness is equivalent to Osher's condition. In terms of the parameters $a, b \in \mathbb{R}$ the strong well-posedness is equivalent to impose that $|ab| < 1$. Which means, in terms of wave packets propagation, that the energy is descreasing after on couple of reflections against the sides $\{x_1 = 0\}$ and $\{x_2 = 0\}$.

However, even if the necessary nature of Osher's corner has not rigorously demonstrated yet, we stress that the compatability condition on the trace $v_{|x_1=0}$, (23) is a rigorous condition for the strong well-posedness of (10). Thus we have the following result:

Theorem 2.1 *Under Assumptions 2.1-2.2, assume that the system* (10) *is strongly well posed in the sense of Definiton 2.1 then the trace of the solution v of* (12) *on $\{x_1 = 0\}$ satisfies* (23).

3 Description of the scheme, definitions and assumptions

3.1 Strong stability of finite difference schemes in the full space

In this paragraph we recall some definition about the strong stability of finite difference scheme approximations for hyperbolic Cauchy problem. Thus we consider the scheme:

$$\begin{cases} U_j^{n+s+1} + \sum_{\sigma=0}^{s} Q^\sigma U_j^{n+\sigma} = 0, & \text{for } j \in \mathbb{Z}^2, n \geq 0, \\ U_j^n = u_{0_j}^n, & \text{for } j \in \mathbb{Z}^2, n \in [\![0, s]\!], \end{cases} \tag{24}$$

where the matrices $Q^\sigma \in \mathbf{M}_{N\times N}(\mathbb{R})$ are given by:

$$Q^\sigma := \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} A^{\sigma,\mu} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2}, \tag{25}$$

with $A^{\sigma,\mu} \in \mathbf{M}_{N\times N}(\mathbb{R})$ and where for $k \in \{1, 2\}$, $\mathbf{T}_k^{\mu_k}$ is the μ_k-shift operator in the direction j_k:

$$\forall u \in \ell^2(\mathbb{Z}^2), (\mathbf{T}_1^{\mu_1} u)_j := u_{j_1+\mu_1, j_2}, \ (\mathbf{T}_2^{\mu_2} u)_j := u_{j_1, j_2+\mu_2}.$$

Note that in (25), $\mu := (\mu_1, \mu_2)$.

The stability of (24) is defined as follows:

Definition 3.1 *The scheme* (24) *is strongly stable if there exists $C > 0$ such that for all $\Delta t \in]0, 1]$, for all initial condition $(u_{0_j}^n)_{n\in[\![0,s]\!], j\in\ell^2(\mathbb{Z}^2)}$ and for all $n \in \mathbb{N}$, we have the estimate:*

$$\sum_{j\in\mathbb{Z}^2} \Delta x |U_j^n|^2 \leq C \sum_{n=0}^{s} \sum_{j\in\mathbb{Z}^2} \Delta x |u_{0_j}^n|^2, \tag{26}$$

where $\Delta x := \Delta x_1 \Delta x_2$.

We introduce the following so-called amplification matrix $\mathscr{A} \in \mathbf{M}_{N(s+1)\times N(s+1)}(\mathbb{R})$ defined for all $\kappa = (\kappa_1, \kappa_2) \in \mathbb{C}^2 \setminus \{0\}$ by:

$$\mathscr{A}(\kappa) := \begin{bmatrix} \widehat{Q^0}(\kappa) & \cdots & \cdots & \widehat{Q^s}(\kappa) \\ I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & I & 0 \end{bmatrix}, \text{ where } \forall \sigma \in [\![0, s]\!],\ \widehat{Q^\sigma}(\kappa) := \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} \kappa_1^{\mu_1} \kappa_2^{\mu_2} A^{\sigma,\mu}. \tag{27}$$

Then we have the following (classical) characterization of the strong stability in terms of the spectrum of the amplification matrix (which is analogous to the hyperbolicity assumption in the continous setting) for the scheme (24):

Proposition 3.1 *The scheme* (24) *is strongly stable in the sense of Definition 3.1 if and only if the amplification matrix $\mathscr{A}(\kappa)$ is power bounded. More precisely, there exists $C > 0$ such that for all $m \in \mathbb{N}$, for all $\xi_1, \xi_2 \in \mathbb{R}$:*

$$|\mathscr{A}(e^{i\xi_1}, e^{i\xi_2})^n| \leq C.$$

In particular if (24) *is strongly stable in the sense of Definition 3.1 then we have the so-called Von-Neumann condition: $\xi_1, \xi_2 \in \mathbb{R}$, $\rho(\mathscr{A}(e^{i\xi_1}, e^{i\xi_2})) \leq 1$, where ρ denotes the spectral radius.*

3.2 Finite difference schemes in the quarter space

Firstly let us define the following matrices: for all $z \in \mathbb{C} \setminus \{0\}$ and $\xi_2 \in \mathbb{R}$,

$$\mathbb{A}_1^{\mu_1}(z, \xi_2) := \delta_{\mu_1,0} - \sum_{\sigma=0}^{s} \sum_{\mu_2=-\ell_2}^{r_2} z^{-(\sigma+1)} e^{i\mu_2\xi_2} A^{\sigma,\mu}, \tag{28}$$

and $\forall \xi_1 \in \mathbb{R}$,

$$\forall \mu_2 \in [\![-\ell_2, r_2]\!],\ \mathbb{A}_2^{\mu_2}(z, \xi_1) := \delta_{\mu_2,0} - \sum_{\sigma=0}^{s} \sum_{\mu_1=-\ell_1}^{r_1} z^{-(\sigma+1)} e^{i\mu_1\xi_1} A^{\sigma,\mu}. \tag{29}$$

The first assumption that we will make upon the finite difference scheme in the quarter space is that both of the sides corresponding to "$\{x_1 = 0\}$" and "$\{x_2 = 0\}$" are non characteristic in the sense that :

Assumption 3.1 *For all couple $(z, \xi_1) \in \overline{\mathscr{U}} \times \mathbb{R}$ (resp. $(z, \xi_2) \in \overline{\mathscr{U}} \times \mathbb{R}$), the matrices $\mathbb{A}_2^{r_2}(z, \xi_1)$ and $\mathbb{A}_2^{-\ell_2}(z, \xi_1)$ (resp. $\mathbb{A}_1^{r_1}(z, \xi_2)$ and $\mathbb{A}_1^{-\ell_1}(z, \xi_2)$) are invertible.*

This assumption has to be understand has a discrete version of the non characteristicity assumption in the continous framework (see Assumption 2.2).

We define the following subsets of $\mathbb{Z}^2$:

$$\begin{aligned} \mathscr{I} &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1, j_2 \geq 1 \right\}, \\ \mathscr{C} &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \in [\![1 - \ell_1, 0]\!], j_2 \in [\![1 - \ell_2, 0]\!] \right\} \\ \mathscr{B}_1 &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \in [\![1 - \ell_1, 0]\!], j_2 \geq 1 \right\} \\ \mathscr{B}_2 &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \geq 1, j_2 \in [\![1 - \ell_2, 0]\!] \right\}, \end{aligned} \tag{30}$$

for two fixed integers ℓ_1 and ℓ_2 corresponding respectively to the stencils of the scheme in the "left" and in the "bottom" directions.

The set $\mathscr{I}$ is a discretization of the interior of the quarter space, $\mathscr{B}_1$ and $\mathscr{B}_2$ are discretizations of the two sides of the boundary of the quarter space and finally $\mathscr{C}$ represents a discretization of the corner. The full set of resolution $\mathscr{R}$ is defined by:

$$\mathscr{R} := \mathscr{I} \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{C} = \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \geq 1 - \ell_1, j_2 \geq 1 - \ell_2 \right\}.$$

In order to state the strong stability estimate for the solution of the finite difference scheme it will be convenient to introduce the extended discretizations of the two sides of the boundary. They are defined by:

$$\begin{aligned} \overline{\mathscr{B}_1} &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \in [\![1 - \ell_1, r_1]\!], j_2 \geq 1 - \ell_2 \right\} \\ \overline{\mathscr{B}_2} &:= \left\{ j = (j_1, j_2) \in \mathbb{Z}^2 | j_1 \geq 1 - \ell_1, j_2 \in [\![1 - \ell_2, r_2]\!] \right\}, \end{aligned} \tag{31}$$

where r_1 and r_2 are two fixed integers that will correspond to the stencils of the scheme in the "right" and in the "top" directions.

The finite difference scheme approximation of (10) that we will consider in this article reads:

$$\begin{cases} U_j^{n+s+1} + \sum_{\sigma=0}^{s} Q^\sigma U_j^{n+\sigma} = \Delta t f_j^{n+s+1}, & \text{for } j \in \mathscr{I}, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_1^{\sigma,j_1} U_{1,j_2}^{n+\sigma} = g_{1,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_1, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} B_2^{\sigma,j_2} U_{j_1,1}^{n+\sigma} = g_{2,j}^{n+s+1}, & \text{for } j \in \mathscr{B}_2, n \geq 0, \\ U_j^{n+s+1} + \sum_{\sigma=0}^{s+1} C^{\sigma,j} U_{1,1}^{n+\sigma} = h_j^{n+s+1}, & \text{for } j \in \mathscr{C}, n \geq 0, \\ U_j^n = 0, & \text{for } j \in \mathscr{R}, n \in [\![0, s]\!], \end{cases} \tag{32}$$

where we recall that the discretizing operators in the interior $Q^\sigma \in \mathbf{M}_{N \times N}(\mathbb{R})$ are defined in (25).

The boundary operators B_1^{σ,j_1}, B_2^{σ,j_2} are defined by:

$$B_1^{\sigma,j_1} := \sum_{\mu_1=0}^{b_{11}} \sum_{\mu_2=0}^{b_{12}} B_1^{\sigma,\mu,j_1} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2}, \text{ and } B_2^{\sigma,j_1} := \sum_{\mu_1=0}^{b_{21}} \sum_{\mu_2=0}^{b_{22}} B_2^{\sigma,\mu,j_2} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2}, \tag{33}$$

for some fixed intergers b_{11}, b_{12}, b_{21} and b_{22} corresponding to the number of space discretization steps for the boundary conditions, and where the matrices B_1^{σ,μ,j_1} and B_2^{σ,μ,j_2} are fixed elements of $\mathbf{M}_N(\mathbb{R})$.

The corner operator is given by:

$$C^{\sigma,j} := \sum_{\mu_1=0}^{c_1} \sum_{\mu_2=0}^{c_2} C_1^{\sigma,\mu,j} \mathbf{T}_1^{\mu_1} \mathbf{T}_2^{\mu_2}, \tag{34}$$

for two fixed integers c_1 and c_2 and where the matrices $C^{\sigma,\mu,j}$ are fixed in $\mathbf{M}_N(\mathbb{R})$.

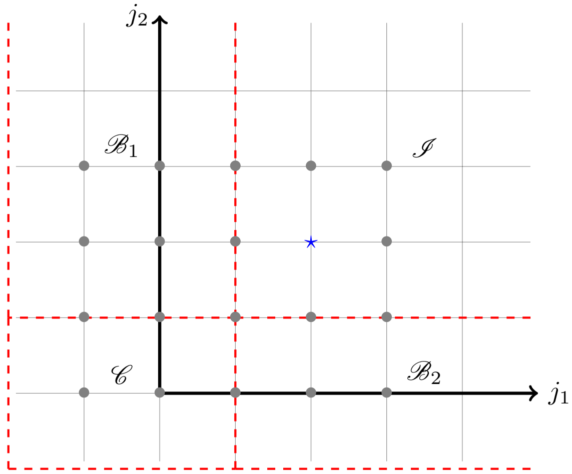

Figure 1: The set of resolution and the dependency set in the interior of $U_{2,2}$ for $\ell_1 = 3$, $\ell_2 = 2$ and $r_1 = r_2 = 1$.

Before we turn to the definition of strong stability for the scheme (32) we give some comments about how this scheme operates. The initial condition gives the value of the U_j^n for $n \in [\![0, s]\!]$, we describe how U_j^{s+1} is computed. The discretization in the interior only involve the U_j^n, for $n \in [\![0, s]\!]$ and $j \in \mathscr{R}$ so its permits to determine the U_j^{s+1} for $j \in \mathscr{I}$. Then the boundary and corner conditions involve the U_j^n, for $n \in [\![0, s]\!]$ but also the U_j^{s+1} for $j \in \mathscr{I}$ which has already been determined. Consequently it permits to determine the values of the U_j^{s+1} for $j \in \mathscr{R} \setminus \mathscr{I}$ and thus complete the determination of (U_j^{s+1}).

Note that with this choice of boundary and corners condition the boundary and corner values only depend on the value of the solution in $\mathscr{I}$ at the time of computation. As mentionned in [**?**] others choices of boundary and corner conditions are possible. For example one can choose to compute the value at the corner in terms of the boundary and interior values. These schemes are of course more complicated and their study is left for future works.

The strong stability definition is the following.

Definition 3.2 (Strong stability) *We say that the finite difference schme approximation* (32) *is strongly stable if there exists a constant $C > 0$ such that for all source terms $(f_j^n) \in \ell^2(\mathscr{I})$, $(g_{1,j}^n) \in \ell^2(\mathscr{B}_1)$, $(g_{2,j}^n) \in \ell^2(\mathscr{B}_2)$ and $(h_j^n) \in \ell^2(\mathscr{C})$ and for all $\Delta t \in]0, 1]$ for all $\gamma > 0$, the solution $(U_j^n)_{j \in \mathscr{R}}$ of* (32) *satisfies*

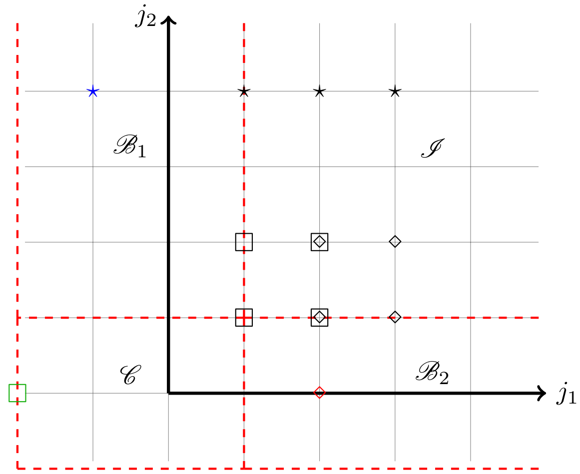

Figure 2: The sets of dependancy at the boundaries and at the corner of $U_{-1,3}$ ($\star$), $U_{2,0}$ ($\diamond$) and $U_{-2,0}$ ($\square$) for $b_{11} = 2$, $b_{12} = 0$, $b_{21} = b_{22} = c_1 = c_2 = 1$

the estimate:

$$\frac{\gamma}{\gamma\Delta t+1}\sum_{n\geq s+1}\Delta t e^{-2\gamma n\Delta t}|||U_j^n|||_{\mathscr{R}}^2 + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}\|U_j^n\|_{\overline{\mathscr{B}_k}}^2 \tag{35}$$
$$\leq C\left(\frac{\gamma\Delta t+1}{\gamma}\sum_{n\geq s+1}\Delta t e^{-2\gamma n\Delta t}|||f_j^n|||_{\mathscr{I}}^2 + \sum_{k=1}^{2}\sum_{n\geq s+1}\Delta t\Delta x_{3-k}e^{-2\gamma n\Delta t}\|g_{k,j}^n\|_{\mathscr{B}_k}^2 + \sum_{n\geq s+1}\Delta t\Delta x_2 e^{-2\gamma n\Delta t}\|h_j^n\|_{\mathscr{C}}^2\right),$$

where for $\mathfrak{I}\subset\mathbb{Z}^2$ the weighted ℓ^2-norm $|||\cdot|||_{\mathfrak{I}}$ is defined by:

$$\forall (U_j)\in\ell^2(\mathfrak{I}),\ |||U|||_{\mathfrak{I}} := \Delta x\sum_{j\in\mathfrak{I}}|U_j|^2.$$

Remark • The estimate (35) is just a natural generalization of the standard energy estimate in the half space geometry used for example in [Cou11].The only changes are that in the left hand side of (35) we ask a control of the two boundaries and that the right hand side involves the source term at the corner $(h_j^n)_{j\in\mathscr{C}}$. In the enrergy estimate in the continous setting, see (11), the right hand side only involves the source terms of the source terms in the interior and on the two sides of the boundary. However in for schemes we have to ask the source term at the corner ohterwise the solution of a totally homogeneous scheme except at the corner should be zero which is clearly not the case.

• Also note that when we take the limit $\Delta t\downarrow 0$ under suitable CFL conditions then we recover (at least formally) the energy estimate for the continous problem that is to say (11). So this estimate seems to be consistent with the litterature at least at the formal level and thus (35) is effectively a discretized version of (11).

• An other possible choice for the boundary control in the left hand side of (35) is of course to choose the norms on $\mathscr{B}_k$, $k\in\{1,2\}$ which leads to a weaker stability definition. However has we will see in () the necessary condition for strong stability, the norms on $\overline{\mathscr{B}_k}$, $k\in\{1,2\}$, are much more natural because they are the quantities involved in the necessary condition for strong stability.

• A last remark is that the choice of the weight in front of the source term at the corner in the right hand side of (35) is totally arbitrary. Indeed from the CFL conditions we have $\Delta x_1 \sim \Delta x_2$.

Without loss of generality we can assume that in (35), $\Delta t = 1$. Consequently from the definition of the CFL numbers we have $\Delta x_j = \frac{1}{\lambda_j}$ and the energy estimate (36) equivalently reads: there exists $C > 0$ such that for all $\gamma > 0$ we have:

$$\begin{aligned} \frac{\gamma}{\gamma+1} \sum_{n \geq s+1} e^{-2\gamma n} \|U_j^n\|_{\mathscr{R}}^2 &+ \sum_{k=1}^{2} \sum_{n \geq s+1} e^{-2\gamma n} \|U_j^n\|_{\overline{\mathscr{B}_k}}^2 \\ &\leq C \left(\frac{\gamma+1}{\gamma} \sum_{n \geq s+1} e^{-2\gamma n} \|f_j^n\|_{\mathscr{I}}^2 + \sum_{k=1}^{2} \sum_{n \geq s+1} e^{-2\gamma n} \|g_{k,j}^n\|_{\mathscr{B}_k}^2 \right. \\ &\left. + \sum_{n \geq s+1} e^{-2\gamma n} \|h_j^n\|_{\mathscr{C}}^2 \right), \end{aligned} \tag{36}$$

that is to say that to check the strong stability it is sufficient to assume $\Delta t = 1$.

As it as been done in Paragraph 2.1, the first step in the study of the strong stability of (32) is to replace the time variable n by a complex parameter $z \in \mathscr{U}$. This is the aim of the following Section.

4 The time-resolvent scheme

In the PDE framework (see Paragraph 2.1) to transform the time derivative in time to a dependency according to a complex parameter it is sufficient to make a Laplace transform in time. This gives the time-resolvent pde. In the discrete setting to obtain the time-resolvent scheme it is sufficient to set:

$$U_j^n := z^n V_j, \; f_j^n := z^n f_j, \; g_{1,j}^n := z^n g_{1,j}, \; g_{2,j}^n := z^n g_{2,j} \text{ and } h_j^n := z^n h_j,$$

where $z \in \mathbb{C} \setminus \{0\}$. The sequence $(V_j)_{j \in \mathscr{R}}$ then formally satisfies the so-called time-resolvent scheme:

$$\begin{cases} V_j + \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^{s-\sigma} V_j = f_j, & \text{for } j \in \mathscr{I}, \\ V_j + \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_1^{s-\sigma, j_1} V_j = g_{1,j}, & \text{for } j \in \mathscr{B}_1, \\ V_j + \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{s-\sigma, j_2} V_j = g_{2,j}, & \text{for } j \in \mathscr{B}_2, \\ V_j + \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} C^{s-\sigma, j} V_j = h_j, & \text{for } j \in \mathscr{C}, \end{cases} \tag{37}$$

and we introduce the following definition for strong stability of the time-resolvent scheme (37):

Definition 4.1 *[Time-resolvent strong stability]The time resolvent scheme* (37) *is said to be strongly stable if there exits a constant $C > 0$ such that for all $z \in \mathscr{U}$, for all source terms $(f_j) \in \ell^2(\mathscr{I})$, $(g_{1,j}) \in \ell^2(\mathscr{B}_1)$, $(g_{2,j}) \in \ell^2(\mathscr{B}_2)$ and $(h_j) \in \ell^2(\mathscr{C})$, the time-resolvent scheme* (37) *admits a unique solution $(V_j) \in \ell^2(\mathscr{R})$ satisfying that there exists $C > 0$ such that for all $z \in \mathscr{U}$ we have:*

$$\frac{|z|-1}{|z|} \sum_{j \in \mathscr{R}} |V_j|^2 + \sum_{k=1}^{2} \sum_{j \in \overline{\mathscr{B}_k}} |V_j|^2 \leq C \left(\frac{|z|}{|z|-1} \sum_{j \in \mathscr{I}} |f_j|^2 + \sum_{k=1}^{2} \sum_{j \in \mathscr{B}_k} |g_{k,j}|^2 + \sum_{j \in \mathscr{C}} |h_j|^2 \right). \tag{38}$$

Once again, we remark that (38) is just a disretized version of the energy estimate for the time-resolvent PDE (13).

As in the continous framework the strong stability for the scheme (32) is equivalent to the strong stability of the time-resolvent scheme (37). More precisely:

Theorem 4.1 ([BGS72]) *The scheme* (32) *is strongly stable in the sense of Definition 3.2 if and only if the time-resolvent scheme* (37) *is strongly stable in the sense of Definition 4.1.*

Proof : We are mainly interested in the statement of a necessary condition for the strong stability of (32) so we will only demonstrate the implication "If the scheme (32) is strongly stable then the time resolvent scheme (37) is strongly stable". The counterpart is a straightforward generalization of the existing result in the half space geometry and can be find in [BGS72] or [Cou13] for example.

The proof described here comes from [Cou13], and is based upon the following lemma:

Lemma 4.1 *For all $m \geq 1$ let ρ_m be given by*

$$\rho_m(\theta) := \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} e^{-ik\theta},$$

then $(\rho_m)_{m\in\mathbb{N}^*}$ satisfies the following properties:

- *ρ_m is 2π-periodic and*
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\rho_m(\theta)|^2 \, d\theta = 1.$$
- *For all $a \in \left]0, \frac{pi}{2}\right]$,*
$$\lim_{n\to+\infty} \int_a^{\pi} |\rho_m(\theta)|^2 \, d\theta = 0.$$
- *For all $H \in \mathscr{C}^0(\mathbb{R})$ such that $\sup_{\theta\in\mathbb{R}}(1+\theta^2)|H(\theta)| < +\infty$, we have:*
$$\lim_{n\to+\infty} \frac{1}{2\pi}\int_{\mathbb{R}} H(\theta)|\rho_m(\theta)|^2 \, d\theta = \sum_{k\in\mathbb{Z}} H(2k\pi).$$

We refer to [Cou13] for a proof of this lemma.

The following proof is very similar to the one given in Paragraph 2.1, except that instead of setting $U_j^n := z^n V_j$, we introduce a truncated in n (and renormalized) sequence $(U_j^n(m))$ that tends to (U_j^n) in ℓ^2 for large values of m. We then write the scheme satisfied by $(U_j^n(m))$ and show that the new source terms also tends to the sequences (f_j), $(g_{k,j})$ and (h_j). We then conclude thanks to the energy estimate (36) assumed on $(U_j^n(m))$ and by taking the limit as $m \uparrow \infty$.

Following [Cou13] we firstly assume that the scheme (37) has a unique solution $(V_j)_{j\in\mathscr{R}} \in \ell^2(\mathscr{R})$ and we show the *a priori* energy estimate (38). To do this we define the sequences: for a fixed $\underline{z} \in \mathscr{U}$,

$$\begin{aligned}
\forall j \in \mathscr{R},\ \forall n \geq 0,\ U_j^n(m) &:= \begin{cases} \frac{\underline{z}}{\sqrt{m}} V_j, \text{ if } s+1 \leq n \leq s+m, \\ 0 \text{ otherwise,} \end{cases} \\
\forall j \in \mathscr{I},\ \forall n \geq s,\ f_j^n(m) &:= U_j^{n+1}(m) - \sum_{\sigma=0}^{s} Q^\sigma U_j^{n-\sigma}(m), \\
\forall j \in \mathscr{B}_1,\ \forall n \geq s+1,\ g_{1,j}^n(m) &:= U_j^{n+1}(m) - \sum_{\sigma=0}^{s} B_1^{\sigma,j_1} U_j^{n-1-\sigma}(m), \\
\forall j \in \mathscr{B}_2,\ \forall n \geq s+1,\ g_{2,j}^n(m) &:= U_j^{n+1}(m) - \sum_{\sigma=0}^{s} B_2^{\sigma,j_2} U_j^{n-1-\sigma}(m), \\
\forall j \in \mathscr{C},\ \forall n \geq s+1,\ h_j^n(m) &:= U_j^{n+1}(m) - \sum_{\sigma=0}^{s} C^{\sigma,j} U_j^{n-1-\sigma}(m).
\end{aligned}$$

By construction the sequence $(U_j^n(m))_{n,j}$ satisfies:

$$\begin{cases} U_j^{n+1}(m) = \sum_{\sigma=0}^{s} Q^\sigma U_j^{n-\sigma}(m) + f_j^n(m), & j \in \mathscr{I}, n \geq s, \\ U_j^{n+1}(m) = \sum_{\sigma=0}^{s} B_1^{\sigma,j_1} U_j^{n-\sigma}(m) + g_{1,j}^n(m), & j \in \mathscr{B}_1, n \geq s, \\ U_j^{n+1}(m) = \sum_{\sigma=0}^{s} B_2^{\sigma,j_2} U_j^{n-\sigma}(m) + g_{2,j}^n(m), & j \in \mathscr{B}_2, n \geq s, \\ U_j^{n+1}(m) = \sum_{\sigma=0}^{s} C^{\sigma,j} U_j^{n-\sigma}(m) + h_j^n(m), & j \in \mathscr{C}, n \geq s, \\ U_j^n(m) = 0, & j \in \mathscr{R}, n \in [\![0,s]\!], \end{cases} \tag{39}$$

so we can apply the energy estimate (35) to the parameter $\underline{\gamma} := \ln|\underline{z}|$. This gives:

$$\begin{aligned} &\frac{\underline{\gamma}}{\underline{\gamma}+1} \sum_{n\geq s+1} \sum_{j\in\mathscr{R}} e^{-2\underline{\gamma}n} |U_j^n(m)|^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} e^{-2\underline{\gamma}n} \sum_{j\in\overline{\mathscr{B}_k}} |U_j^n(m)|^2 \\ &\quad \leq C \left(\frac{\underline{\gamma}+1}{\underline{\gamma}} \sum_{n\geq s} \sum_{j\in\mathscr{I}} e^{-2\underline{\gamma}n} |f_j^n(m)|^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} e^{-2\underline{\gamma}n} \sum_{j\in\mathscr{B}_k} |g_{k,j}^n(m)|^2 + \sum_{n\geq s+1} e^{-2\underline{\gamma}n} \sum_{j\in\mathscr{C}} |h_j^n(m)|^2 \right). \end{aligned}$$

By definition of the sequence $(U_j^n(m))$ we have that for all $j \in \mathscr{R}$:

$$\sum_{n\geq s+1} e^{-2\underline{\gamma}n} |U_j^n(m)|^2 = \sum_{n=s+1}^{s+m} e^{-2\underline{\gamma}n} |\underline{z}|^2 |V_j|^2 = |V_j|^2,$$

so thanks to the inequality $\frac{1-|\underline{z}|}{|\underline{z}|} \leq \frac{\underline{\gamma}+1}{\underline{\gamma}} \leq 2\frac{1-|\underline{z}|}{|\underline{z}|}$, the previous energy estimate becomes:

$$\begin{aligned} &\frac{1-|\underline{z}|}{|\underline{z}|} \sum_{j\in\mathscr{R}} |V_j|^2 + \sum_{k=1}^{2} \sum_{j\in\overline{\mathscr{B}_k}} |V_j^n|^2 \\ &\quad \leq C \left(\frac{|\underline{z}|}{1-|\underline{z}|} \sum_{n\geq s} \sum_{j\in\mathscr{I}} e^{-2\underline{\gamma}n} |f_j^n(m)|^2 + \sum_{k=1}^{2} \sum_{n\geq s+1} e^{-2\underline{\gamma}n} \sum_{j\in\mathscr{B}_k} |g_{k,j}^n(m)|^2 + \sum_{n\geq s+1} e^{-2\underline{\gamma}n} \sum_{j\in\mathscr{C}} |h_j^n(m)|^2 \right), \end{aligned}$$

and to establish (38) it remains to pass to the limit $m \uparrow \infty$ in the right hand side. We will here only justify the limit for the term involving $(f_j^n(m))$ the other limits follows exactly the same arguments.

We define the following step functions on $\mathbb{R}_+$:

$$U_j(m,t) := \begin{cases} 0, \text{ if } t \in [0, s+1[, \\ U_j^n(m) \text{ if } t \in [n, n+1[, n \geq s+1, \end{cases} \qquad f_j(m,t) := \begin{cases} 0, \text{ if } t \in [0, s[, \\ f_j^n(m) \text{ if } t \in [n, n+1[, n \geq s, \end{cases}$$

$$g_{1,j}(m,t) := \begin{cases} 0, \text{ if } t \in [0, s+1[, \\ g_{1,j}^n(m) \text{ if } t \in [n, n+1[, n \geq s+1, \end{cases} \qquad g_{2,j}(m,t) := \begin{cases} 0, \text{ if } t \in [0, s+1[, \\ g_{2,j}^n(m) \text{ if } t \in [n, n+1[, n \geq s+1, \end{cases}$$

$$h_j(m,t) := \begin{cases} 0, \text{ if } t \in [0, s+1[, \\ h_j^n(m) \text{ if } t \in [n, n+1[, n \geq s+1. \end{cases}$$

In terms of these functions, the scheme (39) is equivalent to the continous in time, discretized in space scheme:

$$\begin{cases} U_j(m,t+1) = \sum_{\sigma=0}^{s} Q^\sigma U_j(m,t-\sigma) + f_j(m,t), & j \in \mathscr{I}, t \geq s, \\ U_j(m,t+1) = \sum_{\sigma=0}^{s} B_1^{\sigma,j_1} U_j(m,t-\sigma) + g_{1,j}(m,t), & j \in \mathscr{B}_1, t \geq s, \\ U_j(m,t+1) = \sum_{\sigma=0}^{s} B_2^{\sigma,j_2} U_j(m,t-\sigma) + g_{2,j}(m,t), & j \in \mathscr{B}_2, t \geq s, \\ U_j^{n+1}(m,t+1) = \sum_{\sigma=0}^{s} C^{\sigma,j} U_j(m,t-\sigma) + h_j(m,t), & j \in \mathscr{C}, t \geq s. \end{cases} \tag{40}$$

We take the Laplace transform of $U_j(m,t)$ in the above equations to obtain that:

$$\begin{cases} \widehat{U}_j(m,\tau) = \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma \widehat{U}_j(m,\tau) + z^{-1}\widehat{f}_j(m,\tau), & j \in \mathscr{I}, \\ \widehat{U}_j(m,\tau) = \sum_{\sigma=0}^{s} z^{-(\sigma+1)} B_1^{\sigma,j_1} \widehat{U}_j(m,\tau) + \widehat{g}_{1,j}^n(m,\tau), & j \in \mathscr{B}_1,, \\ \widehat{U}_j(m,\tau) = \sum_{\sigma=0}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} \widehat{U}_j(m,\tau) + \widehat{g}_{2,j}(m,\tau), & j \in \mathscr{B}_2, \\ \widehat{U}_j(m,\tau) = \sum_{\sigma=0}^{s} z^{-(\sigma+1)} C^{\sigma,j} \widehat{U}_j(m,\tau) + \widehat{h}_j(m,\tau), & j \in \mathscr{C}, \end{cases} \tag{41}$$

where τ is the dual variable of t and where we used the notation $z := e^{\tau}$. We can also compute explicitly:

$$\forall \theta \in \mathbb{R},\ \widehat{U}_j(m, \underline{\tau} + i\theta) = \frac{1 - \underline{z}^{-1} e^{-i\theta}}{\underline{\tau} + i\theta} e^{-i(s+1)\theta} \rho_m(\theta) V_j.$$

Then the first equation of (41) gives:

$$\underline{z}^{-1} e^{-i\theta} \widehat{f}_j(m, \underline{\tau} + i\theta) = \frac{1 - \underline{z}^{-1} e^{-i\theta}}{\underline{\tau} + i\theta} e^{-i(s+1)\theta} \rho_m(\theta) \left(V_j - \sum_{\sigma=0}^{s} \underline{z}^{-(\sigma+1)} e^{-i(\sigma+1)\theta} Q^{\sigma} V_j \right),$$

from which we deduce thanks to Fubini's and Plancherel's theorems that:

$$\begin{aligned}
\sum_{n \geq s} \sum_{j \in \mathscr{I}} \frac{e^{-2\underline{\gamma} n}}{|\underline{z}|} |f_j^n(m)|^2 &= \frac{2\underline{\gamma}}{1 - e^{-2\underline{\gamma}}} \sum_{j \in \mathscr{I}} |\underline{z}|^{-2} \int_{\mathbb{R}_+} e^{-2\underline{\gamma} t} |f_j(m,t)|^2 \, dt \\
&= \frac{2\underline{\gamma}}{1 - e^{-2\underline{\gamma}}} \sum_{j \in \mathscr{I}} |\underline{z}|^{-2} \int_{\mathbb{R}} |\widehat{f}_j(m, \underline{\tau} + i\theta)|^2 \, d\theta \\
&= \frac{1}{2\pi} \frac{2\underline{\gamma}}{1 - e^{-2\underline{\gamma}}} \int_{\mathbb{R}} H(\theta) |\rho_m(\theta)|^2 d\theta, \qquad (42)
\end{aligned}$$

where we used the notation:

$$H(\theta) := \left| \frac{1 - \underline{z}^{-1} e^{-i\theta}}{\underline{\tau + i\theta}} \right|^2 \sum_{j \in \mathscr{I}} \left| V_j - \sum_{\sigma=0}^{s} \underline{z}^{-1} e^{-i(\sigma+1)\theta} Q^{\sigma} W_j \right|^2.$$

The function H satisfies the properties $i) - iii)$ of Lemma 4.1. So we deduce that: () la limite

$$\sum_{n \geq s} \sum_{j \in \mathscr{I}} \frac{e^{-2\underline{\gamma} n}}{|\underline{z}|} |f_j^n(m)|^2 = \frac{2\underline{\gamma}}{1 - e^{-2\underline{\gamma}}} \sum_{k \in \mathbb{Z}} \left| \frac{1 - \underline{z}^{-1}}{\underline{\tau} + 2ik\pi} \right|^2 \sum_{j \in \mathscr{I}} |(L(\underline{z})V)_j|^2,$$

where we defined:

$$\forall z \in \mathscr{U},\ V \in \ell^2(\mathscr{R}), (L(\underline{z})V)_j := \left\{ \begin{array}{l} V_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^{\sigma} V_j, \text{ for } j \in \mathscr{I}, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_1^{\sigma, j_1} V_{1, j_2}, \text{ for } j \in \mathscr{B}_1, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma, j_2} V_{j_1, 1}, \text{ for } j \in \mathscr{B}_2, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} C^{\sigma, j} V_{1,1}, \text{ for } j \in \mathscr{C}, \end{array} \right. . \qquad (43)$$

To conclude we use the formula:

$$\frac{2\underline{\gamma}}{1 - e^{-2\underline{\gamma}}} \sum_{k \in \mathbb{Z}} \left| \frac{1 - \underline{z}^{-1}}{\underline{\tau} + 2ik\pi} \right|^2 = 1,$$

whose proof can be found in [Cou13].

This complete the proof of the energy estimate (38), it remains to show the existence of a solution $(V_j)_{j \in \mathscr{R}}$ of the scheme (37). As in [Cou13] the existence of $(V_j)_{j \in \mathscr{R}}$ is a consequence of the two following lemmas:

Lemma 4.2 *There exists $R \geq 1$ such that for all $z \in \mathbb{C}$ satisfying $|z| \geq R$ the operator $(L(z)$ defined by (43) is invertible on $\ell^2(\mathscr{R})$.*

Proof : We define:

$$L_\infty : \ell^2(\mathscr{R}) \to \ell^2(\mathscr{R}), \text{ by } (L_\infty V)_j := \begin{cases} V_j, & j \in \mathscr{I}, \\ V_j - B_1^{-1, j_1} V_{1, j_2}, & j \in \mathscr{B}_1, \\ V_j - B_2^{-1, j_2} V_{j_1, 1}, & j \in \mathscr{B}_2, \\ V_j - C^{-1, j} V_{1,1}, & j \in \mathscr{C}, \end{cases}.$$

Clearly L_∞ is a bounded invertible operator on $\ell^2(\mathscr{R})$. Moreover we have:

$$z((L_\infty - L(\underline{z}))V)_j := \begin{cases} \sum_{\sigma=0}^{s} z^{-\sigma} Q^\sigma V_j, \text{ for } j \in \mathscr{I}, \\ \sum_{\sigma=0}^{s} z^{-\sigma} B_1^{\sigma,j_1} V_{1,j_2}, \text{ for } j \in \mathscr{B}_1, \\ \sum_{\sigma=0}^{s} z^{-\sigma+1} B_2^{\sigma,j_2} V_{j_1,1}, \text{ for } j \in \mathscr{B}_2, \\ \sum_{\sigma=0}^{s} z^{-\sigma+1} C^{\sigma,j} V_{1,1}, \text{ for } j \in \mathscr{C}, \end{cases},$$

so there exists $C > 0$ such that:

Lemma 4.3 *Let X be a Banach space, E be a nonempty connected set. Let L be a continous function on E with values in the space of bounded operators on X, $\mathcal{B}(X)$. We assume that the following conditions are fulfilled:*

- *there exists $C > 0$ such that for all $e \in E$ and for all $x \in X$, $\|x\|_E \leq C\|L(e)x\|_E$,*
- *there exists $\underline{e} \in E$ such that $L(\underline{e})$ is an isomorphism.*

Then $L(e)$ is an isomorphism for all $e \in E$.

We refer to [[Cou13], Lemma 12] for a proof of Lemma 4.3.

An important corollary of Theorem 4.1 is that as for finite difference schemes approximation in the half space we recover the fact that the so-called Godunov-Ryabenkii condition is necessary for the strong stability of the scheme. More precisely this conditions is the following:

Corollary 4.1 (Godunov-Ryabenkii condition) *Assume that the scheme* (37) *is strongly stable in the sense of Definition 4.1 then for all $z \in \mathscr{U}$ the only solution $V \in \ell^2(\mathscr{R})$ of the homogeneous scheme:*

$$\begin{cases} V_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma V_j = 0, & \text{for } j \in \mathscr{I}, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_1^{\sigma,j_1} V_j = 0, & \text{for } j \in \mathscr{B}_1, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} V_j = 0, & \text{for } j \in \mathscr{B}_2, \\ V_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} C^{\sigma,j} V_j = 0, & \text{for } j \in \mathscr{C}, \end{cases} \tag{44}$$

is zero.

5 The extended time-resolvent scheme

As it as been done in Section 2, once the time variable has been replace by the complex parameter z, the next step of the analysis is to extend the solution by zero for negative values of one of the space variable and to consider a scheme set in the half space. In this section we describe this extension and its influence on the error terms that it induces in the interior and on the boundary.

We have already remarked in Section 2 that the extension by zero for negative x_1 induces an error source term in the interior, namely $\delta_{|x_1=0} u_{|x_1=0}$. In the discrete setting extending by zero the solution of the scheme for $j_1 \leq 1 - \ell_1$ will also induce an error source term in the interior (localized in the strip $[\![1-\ell_1-r_1, -\ell_1]\!] \times [\![1, +\infty[\![$) and it can be show that this error source terms depends on the V_j for $j \in \overline{\mathscr{B}_1}$.
However, and it a new fact compared to the continous setting, the extension will also create an error term on the boundary. To ensure that the boundary error term depends on the V_j for $j \in \overline{\mathscr{B}_1}$, a restriction upon the value of the j_1-stencil in the boundary condition on $\mathscr{B}_2$, namely b_{21} will be necessary.

Let $(W_j)_{j_1 \in \mathbb{Z}, j_2 \geq 1-\ell_2}$ be the extension of $(V_j)_{j \in \mathscr{R}}$, the solution of (37), by zero for $j_1 \leq 1 - \ell_1$. In this section we are interested in the scheme in the half space solved by the sequence (W_j).

First let us consider the interior of the half space $\{j_1 \in \mathbb{Z}, j_2 \geq 1 - \ell_2\}$, that is to say that in the following paragraph we assume that $j_2 \geq 1$. Secondly we will treat the case of the boundary that is to say $j_2 \in [\![1-\ell_2, 0]\!]$. The last paragraph of this section is a summary of the error terms in the interior and on the boundary and contains the study of some particular cases for which the expressions of these errors terms are slightly simpler.

5.1 The error term in the interior

We separate several cases depending on the value of the parameter j_1.

$\diamond$ $j_1 \geq 1$.
By definition of the operators Q^σ, $((Q^\sigma W)_j)_{j\in\mathscr{I}}$ only involves the $(W_j)_{j\in\mathscr{R}}$. But the restriction $(W_j)_{j\in\mathscr{R}}$ is equal to $(V_j)_{j\in\mathscr{R}}$ for $j_1, j_2 \geq 1$ it follows that:

$$W_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma W_j = V_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma V_j = f_j, \text{ for } j \in \mathscr{I}. \tag{45}$$

$\diamond$ $j_1 \leq -(\ell_1 + r_1)$.
Once again by definition of the operators Q^σ it follows that for $j_1 \leq -(\ell_1 + r_1)$, all the terms in the sum $((Q^\sigma W)_j)_{j_1 \leq -(\ell_1+r_1)}$ are zero because the right stencil r_1 is not large enough to catch the W_j for $j_1 \geq 1-\ell_1$. Consequently for $j_1 \leq -(\ell_1 + r_1)$, (W_j) satisfies the interior discretization:

$$W_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma W_j = 0, \quad \text{for } (j_1, j_2) \in]\!] -\infty, -(\ell_1 + r_1)]\!] \times [\![1, +\infty[\![. \tag{46}$$

Finally we consider the last case to study.
$\diamond$ $j_1 \in [\![1 - \ell_1 - r_1, 0]\!]$.
For such values of j_1, the stencil r_1 is large enough to catch some non trivial values of the sequence (W_j). Some tedious, but not difficult, computations show that (W_j) satisfies:

$$W_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma W_j = (L_1(V))_j, \text{ for } (j_1, j_2) \in [\![1 - \ell_1 - r_1, 0]\!] \times [\![1, +\infty[\![. \tag{47}$$

where the sequence $(L_1 V)_j$ is defined by:

$$(L_1(V))_j = (L_1(z, V))_j := \mathbf{1}_{j_1 \geq 1-\ell_1} V_{j_1,j_2} - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} \sum_{\mu_1=1-\ell_1-j_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} A^{\sigma,\mu} V_{\mu_1-j_1, j_2+\mu_2}, \tag{48}$$

where $\mathbf{1}_X$ denotes the characteristic function of the set X.

An important remark for what follows is that as $(L_1 V)_j$ is defined on the strip $[\![1 - \ell_1 - r_1, 0]\!] \times [\![1, \infty[\![$, $(L_1 V)_j$ only involves the V_j for $j \in \overline{\mathscr{B}_1}$. This observation will lead us to the compatibility condition mentioned in the introduction.

We now treat the error term induced by the extension on the boundary, that is to say that in the next paragraph we will assume that $j_2 \in [\![1 - \ell_2, 0]\!]$.

5.2 The error term on the boundary

As it as been done for the error term in the interior, we have to distinguish three cases depending on the value of j_1.

$\diamond$ $j_1 \geq 1 - \ell_1$
From the definition of B_2^{σ,j_2}, the term $B_2^{\sigma,j_2} W_{j_1,1}$ defining the boundary condition only involves the W_{j_1,j_2} for $j_1 \geq 1 - \ell_1$ and $j_2 \geq 1$. For these terms, by definition of (W_j), we deduce that (W_j) satisfies:

$$W_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} W_{j_1,1} = g_{2,j}, \quad \text{for } j \in \mathscr{B}_2 \cup \mathscr{C}. \tag{49}$$

$\diamond$ $j_1 \leq -\ell_1 - b_{21}$
In the framework the boundary condition can not capture non trivial values of $(W_j)_j$ and, as for the interior term, we have a homogeneous boundary condition:

$$W_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} W_{j_1,1} = g_{2,j}, \quad \text{for } j \in]\!] -\infty, -\ell_1 - b_{21}]\!] \times [\![1-\ell_2, 0]\!], \tag{50}$$

$\diamond$ $j_1 \in [\![1-\ell_1-b_{21}, -\ell_1]\!]$
In this framework, some computations show that the extension (W_j) satisfies:

$$W_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} W_{j_1,1} = (K_1(V))_j, \quad \text{for } j \in [\![1-\ell_1-b_{21}, -\ell_1]\!] \times [\![1-\ell_2, 0]\!], \tag{51}$$

where the sequence $(K_1 V)_j$ is defined by

$$(K_1(V))_j = (K_1(z,V))_j := -\sum_{\sigma=0}^{s} z^{-(\sigma+1)} \sum_{\mu_1=1-\ell_1-j_1}^{b_{21}} \sum_{\mu_2=0}^{b_{22}} B_2^{\sigma,j_2,\mu} V_{j_1+\mu_1, j_2+\mu_2}. \tag{52}$$

5.3 Summary and particular cases

We combine (45)-(46)-(47) and (49)-(50)-(51) to obtain that the extension $(W_j)_{j\in\mathbb{Z}\times[\![1-\ell_2,+\infty[\![}$ is solution of the finite difference scheme approximation in the half space:

$$\begin{cases} W_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma W_j = \mathcal{F}_2(f,V)_j, & \text{for } j_1 \in \mathbb{Z}, j_2 \geq 1, \\ W_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_2^{\sigma,j_2} W_{j_1,1} = \mathcal{G}_2(g_2,V)_j, & \text{for } j_1 \in \mathbb{Z}, j_2 \in [\![1-\ell_2, 0]\!], \end{cases}, \tag{53}$$

where the source terms $\mathcal{F}_2(f,V)$ and $\mathcal{G}_2(g_2,V)$ are defined by:

$$\forall j_2 \geq 1,\ \mathcal{F}_2(f,V) = \mathcal{F}_2(z,f,V) = \ := \begin{cases} f_j, & \text{if } j_1 \geq 1, \\ (L_1(V))_j, & \text{if } j_1 \in [\![1-\ell_1-r_1, 0]\!], \\ 0, & \text{if } j_1 \leq -(\ell_1+r_1), \end{cases} \tag{54}$$

$$\text{and } \forall j_2 \in [\![1-\ell_2, 0]\!],\ \mathcal{G}_2(g_2,V) = \mathcal{G}_2(z,g_2,V) \ := \begin{cases} g_{2,j}, & \text{if } j_1 \geq 1-\ell_1, \\ (K_1(V))_j, & \text{if } j_1 \in [\![1-\ell_1-b_{21}, -\ell_1]\!], \\ 0, & \text{if } j_1 \leq -(\ell_1+b_{21}). \end{cases} \tag{55}$$

Moreover from the energy estimate (38), the sequence $(W_j) \in \ell^2(\mathbb{Z} \times [\![1-\ell_2, +\infty[\![)$ and from the uniqueness of (V_j) it is uniquely determined. We sum up this Section by the following proposition:

Proposition 5.1 *The extension by zero for $j_1 \leq 1-\ell_1$ of (V_j) the solution of the time-resolvent scheme* (37) *defined a unique element of $\ell^2(\mathbb{Z} \times [\![1-\ell_2, +\infty[\![)$. Moreover, the extension (W_j) satisfies the scheme* (53) *in the half space $\mathbb{Z} \times [\![1-\ell_2, +\infty[\![$.*

Similarly if we define $(\widetilde{W}_j)_j$ as the extension of $(V_j)_j$ by zero for $j_2 \leq 1-\ell_2$, then the same computations show that the sequence is uniquely determined $(\widetilde{W}_j)_j$ in $\ell^2([\![1-\ell_1, +\infty[\![\times \mathbb{Z})$ and satisfies the scheme in the half space $[\![1-\ell_1, +\infty[\![\times \mathbb{Z}$:

$$\begin{cases} \widetilde{W}_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma \widetilde{W}_j = \mathcal{F}_1(f,V)_j, & \text{for } j_1 \geq 1, j_2 \in \mathbb{Z}, \\ \widetilde{W}_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_1^{\sigma,j_1} \widetilde{W}_{1,j_2} = \mathcal{G}_1(g_1,V)_j, & \text{for } j_1 \in [\![1-\ell_1, 0]\!], j_2 \in \mathbb{Z}, \end{cases}, \tag{56}$$

where the source terms $\mathcal{F}_1(f,V)$ and $\mathcal{G}_1(g_1,V)$ are defined by:

$$\forall j_1 \geq 1,\ \mathcal{F}_1(f,V) = \mathcal{F}_1(z,f,V) = \ := \begin{cases} f_j, & \text{if } j_2 \geq 1, \\ L_1(V)_j, & \text{if } j_2 \in [\![1-\ell_2-r_2, 0]\!], \\ 0, & \text{if } j_2 \leq -(\ell_2+r_2), \end{cases} \tag{57}$$

$$\text{and } \forall j_1 \in [\![1-\ell_1, 0]\!],\ \mathcal{G}_1(g_1,V) = \mathcal{G}_1(z,g_1,V) \ := \begin{cases} g_{1,j}, & \text{if } j_2 \geq 1-\ell_2, \\ K_1(V)_j, & \text{if } j_2 \in [\![1-\ell_2-b_{12}, -\ell_2]\!], \\ 0, & \text{if } j_2 \leq -(\ell_2+b_{12}). \end{cases} \tag{58}$$

and where the sequences $(L_1(V))_j$ and $(K_1(V))_j$ are given by:

$$(L_1(V))_j = (L_1(z,V))_j := \mathbf{1}_{j_2\geq 1-\ell_2} V_{j_1,j_2} - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} \sum_{\mu_2=1-\ell_2-j_2}^{r_2} \sum_{\mu_1=-\ell_1}^{r_1} A^{\sigma,\mu} V_{\mu_1+j_1,\mu_2-j_2}, \tag{59}$$

and

$$(K_1(V))_j = (K_1(z,V))_j := -\sum_{\sigma=0}^{s} z^{-(\sigma+1)} \sum_{\mu_2=1-\ell_2-j_2}^{b_{12}} \sum_{\mu_1=0}^{b_{11}} B_1^{\sigma,j_1,\mu} V_{j_1+\mu_1,j_2+\mu_2}. \tag{60}$$

Remark
- We remark that in the particular case $b_{21}=0$ (resp. $b_{12}=0$) then the error term on the boundary $(\mathcal{G}_1(V))_j$ (resp. $(\mathcal{G}_2(V))_j$) reduces to:

$$\begin{cases} g_{2,j}, & \text{if } j_1 \geq 1-\ell_1, \\ 0, & \text{if } j_1 \leq -\ell_1, \end{cases},\ \left(\text{resp.} \begin{cases} g_{1,j}, & \text{if } j_2 \geq 1-\ell_2, \\ 0, & \text{if } j_2 \leq -\ell_2, \end{cases}\right)$$

  which is just the extension of $(g_{2,j})_j$ (resp. $(g_{1,j})_j$) by zero for $j_1 \leq -\ell_1$ (resp. $j_2 \leq -\ell_2$). In that particular case we do not have any error on the boundary $\mathscr{B}_2$ (resp. $\mathscr{B}_1$).

- If $b_{21} \leq 1+\ell_1+r_1$ (resp. $b_{12} \leq 1+\ell_2+r_2$) then the error term on the boundary $(\mathcal{G}_1(V))_j$ (resp. $(\mathcal{G}_2(V))_j$) only involves terms the V_j for $j \in \overline{\mathscr{B}_1}$ (resp. $j \in \overline{\mathscr{B}_2}$).

6 The necessary condition for strong stability

In this section we restrict our attention to a time resolvent finite difference scheme of the form (37) with only a non trivial source term on the boundary $\mathscr{B}_1$, that is to say that we assume $f_j \equiv 0$, $g_{2,j} \equiv 0$ and $h_j \equiv 0$. The associated extended scheme by zero for $j_1 < 1-\ell_1$ reads:

$$\begin{cases} \widetilde{W}_j - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} Q^\sigma \widetilde{W}_j = \mathcal{F}_2(V)_j, & \text{for } j_1 \geq 1, j_2 \in \mathbb{Z}, \\ \widetilde{W}_j - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} B_1^{\sigma,j_1} \widetilde{W}_{1,j_2} = \mathcal{G}_2(V)_j, & \text{for } j_1 \in [\![1-\ell_1, 0]\!], j_2 \in \mathbb{Z}, \end{cases}, \tag{61}$$

where according to (54) and (55) the source terms in (61) are given by:

$$\mathcal{F}_2(V)_j = L_2(V)_j \mathbf{1}_{[\![1-\ell_1-r_1,0]\!]}(j_1), \text{ and } \mathcal{G}_2(V)_j = K_2(V)_j \mathbf{1}_{[\![1-\ell_1-b_{21},-\ell_1]\!]}(j_1).$$

As it has been done in Paragraph 2 in the PDE framework, the last step to obtain the necessary condition for strong stability is to use Fourier transform to study a scheme set in the half space $\mathbb{Z} \times [\![1-\ell_2, \infty[\![$ where after the Fourier transform the variable j_1 acts like a parameter. To perform Fourier transform we define the step function associated to the sequence (W_j) defined by:

$$W_{j_2}(x_1) := W_j \text{ for } x_1 \in [j_1, j_1+1[\,. \tag{62}$$

We also define the steps functions associated to the source terms $(\mathcal{F}_2(V)_j)_j$ and $(\mathcal{G}_2(V)_j)_j$ by:

$$(\mathcal{F}_{2,j_2}(V))(x_1) := \mathcal{F}_2(V)_j \text{ for } x_1 \in [j_1, j_1+1[\text{ and } (\mathcal{G}_{2,j_2}(V))(x_1) := \mathcal{G}_2(V)_j \text{ for } x_1 \in [j_1, j_1+1[\,. \tag{63}$$

In terms of the variable x_1 the scheme (53) equivalently reads: $\forall x_1 \in \mathbb{R}$,

$$\begin{cases} W_{j_2}(x_1) - \sum_{\sigma=0}^{s} z^{-(\sigma+1)} \sum_{\mu_1=-\ell_1}^{r_1} \sum_{\mu_2=-\ell_2}^{r_2} A^{\sigma,\mu} W_{j_2+\mu_2}(x_1-\mu_1) = (\mathcal{F}_{2,j_2}(V))(x_1), \text{ for } j_2 \geq 1, \\ W_{j_2}(x_1) - \sum_{\sigma=-1}^{s} z^{-(\sigma+1)} \sum_{\mu_1=0}^{b_{21}} \sum_{\mu_2=0}^{b_{22}} B_2^{\sigma,\mu,j_2} W_{1+\mu_2}(x_1-\mu_1) = (\mathcal{G}_{2,j_2}(V))(x_1), \text{ for } j_2 \in [\![1-\ell_2, 0]\!]. \end{cases} \tag{64}$$

As $(W_j) \in \ell^2(\mathbb{Z} \times [\![1-\ell_2, \infty[\![)$, the function $W_{j_2} \in L^2(\mathbb{R})$ for all $j_2 \in [\![1-\ell_2, \infty[\![$ so we can perform a Fourier transform in the x_1 variable. Let ξ_1 be the associated dual variable. So the scheme (64) becomes:

$$\begin{cases} \sum_{\mu_2=-\ell_2}^{r_2} \mathbb{A}_2^{\mu_2}(z,\xi_1) \widehat{W}_{j_2+\mu_2}(\xi_1) = (\widehat{\mathcal{F}_{2,j_2}(V)})(\xi_1), & \text{for } j_2 \geq 1, \\ \widehat{W}_{j_2}(\xi_1) - \sum_{\mu_2=0}^{b_{22}} \mathbb{B}_2^{\mu_2,j_2}(z,\xi_1) \widehat{W}_{1+\mu_2}(\xi_1) = (\widehat{\mathcal{G}_{2,j_2}(V)})(\xi_1), & \text{for } j_2 \in [\![1-\ell_2, 0]\!], \end{cases} \tag{65}$$

where we recall that the matrices $\mathbb{A}_2^{\mu_2}$ are defined in (29) and where we set

$$\forall \mu_2 \in [\![0, b_{22}]\!], \forall j_2 \in [\![1-\ell_2, 0]\!],\ \mathbb{B}_2^{\mu_2,j_2}(z,\xi_1) := \sum_{\sigma=-1}^{s} \sum_{\mu_1=0}^{b_{21}} z^{-(\sigma+1)} e^{ij_1\xi_1} B_2^{\sigma,\mu,j_2}. \tag{66}$$

The next paragraph is devoted to a classical reformulation of the scheme (65) in terms of an augmented vector. This reformulation is much more convenient to study because it permits, on the one hand, to give a simple formulation of the uniform Kreiss-Lopatinskii condition and on the other hand to give a simple expression of the solution of (65) given by a discrete Duhamel's formula.

6.1 Reformulation of the j_2-totally resolvent scheme

Let $X_{j_2} = X_{j_2}(\xi_1) := \widehat{W}_{j_2}(\xi_1)$ it will be convenient to express the scheme (65) in terms of an augmented vector $\mathscr{X}_{j_2} := (X_{j_2+r_2-1}, ...X_{j_2-\ell_2}) \in \mathbb{C}^{N(\ell_2+r_2)}$. To do this we have to separate two cases depending on the respective values of b_{22} and r_2.

$\diamond$ The case $b_{22} \leq r_2$. Under a such restriction upon the values of the parameters b_{22} and r_2, the terms appearing in the boundary condition on $\mathscr{B}_2$ are include in the augmented vector $\mathscr{X}_{j_2}$.

Let $\mathbb{M}_2(z) \in \mathbf{M}_{N(\ell_2+r_2)}(\mathbb{C})$ and $\mathbb{B}_2(z) \in \mathbf{M}_{N(\ell_2+r_2)}(\mathbb{C})$ be defined on some neighborhood of $\overline{\mathscr{U}} \times \mathbb{R}$ by the equations:

$$\mathbb{M}_2(\zeta) = \mathbb{M}_2(z,\xi_1) := \begin{bmatrix} -(\mathbb{A}_2^{r_2}(z,\xi_1))^{-1}\mathbb{A}_2^{r_2-1}(z,\xi_1) & \dots & \dots & -(\mathbb{A}_2^{r_2}(z,\xi_1))^{-1}\mathbb{A}_2^{-\ell_2}(z,\xi_1) \\ I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & I & 0 \end{bmatrix}, \tag{67}$$

$$\mathbb{B}_2(\zeta) = \mathbb{B}_2(z,\xi_1) := \begin{bmatrix} 0 & \cdots & 0 & -\mathbb{B}_2^{b_{22},0} & & -\mathbb{B}_2^{0,0} & I & & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & -\mathbb{B}_2^{b_{22},1-\ell_2} & & -\mathbb{B}_2^{0,1-\ell_2} & 0 & & I \end{bmatrix}. \tag{68}$$

Then it is rather easy to see that in terms of the matrices $\mathbb{M}_2(z,\xi_1)$, $\mathbb{B}_2(z,\xi_1)$ and the extended vector $\mathscr{X}_{j_2}$ the scheme (65) becomes:

$$\begin{cases} \mathscr{X}_{j_2+1} = \mathbb{M}_2(\zeta)\mathscr{X}_{j_2} + \mathscr{F}_{2,j_2}(V), & \text{for } j_2 \geq 1, \\ \mathbb{B}_2(\zeta)\mathscr{X}_1 = \mathscr{G}_2(V), \end{cases} \tag{69}$$

where the source terms are given by:

$$\begin{aligned} \mathscr{F}_{2,j_2}(V) &= \mathscr{F}_{2,j_2}(\zeta, V) := (\mathscr{F}_{2,j_2}(V))(z,\xi_1) := (-(\mathbb{A}_2^{r_2}(z,\xi_1))^{-1}(\widehat{\mathcal{F}_{2,j_2}(V)})(\xi_1), 0, \dots, 0) \\ \mathscr{G}_2(V) &= \mathscr{G}_2(\zeta, V) := (\mathscr{G}_2(V))(z,\xi_1) := ((\widehat{\mathcal{G}_{2,0}(V)})(\xi_1), \dots, (\widehat{\mathcal{G}_{2,1-\ell_2}(V)})(\xi_1)). \end{aligned} \tag{70}$$

6.2 The trace operators

In order to solve (69) it is necessary to study the eigenvalues of $\mathbb{M}_2(\zeta)$. The following result is due to () and states that the matrix $\mathbb{M}(\zeta)$ does not have any eigenvalues on the unit circle $\mathbb{S}^2$. In particular, we have a direct decomposition of the space $\mathbb{C}^{N(\ell_2+r_2)}$ in terms of a stable subspace (the generalized eigenspace associated to eigenvalues in $\mathbb{D}$) and an unstable subspace (the generalized eigenspace associated to eigenvalues in $\mathscr{U}$). This result is analogous to Hersh's lemma [Her63] in the continous setting.

Lemma 6.1 *[Eigenvalues of $\mathbb{M}_2(\zeta)$]Under Assumption 3.1, we assume that the discretization for the Cauchy problem is strongly stable in the sense of Definition 3.1. Then for all $z \in \mathscr{U}$, the eigenvalues κ of the matrix $\mathbb{M}_2(\zeta)$ are away from $\mathbb{S}^1$. They are non-zero and are characterized by the relation:*

$$\det(\mathscr{A}(e^{i\xi_1}, \kappa) - zI) = 0,$$

where $\mathscr{A}(\kappa_1, \kappa_2)$ is the amplification matrix defined in (27).

Moreover, there are $N\ell_2$ (counted with multiplicity) stable eigenvalues in $\mathbb{D}$.

The generalized eigenspace associated to the eigenvalues of $\mathbb{M}_2(\zeta)$ in $\mathbb{D}$ is denoted by $E_2^s(\zeta)$, the one associated to the eigenvalues in $\mathscr{U}$ is denoted by $E_2^u(\zeta)$. Thus we have:

$$\forall z \in \mathscr{U}, \forall \xi_1 \in \mathbb{R},\ \mathbb{C}^{N(\ell_2+r_2)} = E_2^s(\zeta) \oplus E_2^u(\zeta). \tag{71}$$

Proof : This proof is a slight modification of the one given in [Cou13] for the geometry of the half line. However we recall it briefly for a sake of completness.

Firstly let us show that the eigenvalues of $\mathbb{M}_2(\zeta)$ are non-zero. Let $Y = (Y_1, \dots, Y_{\ell_2+r_2}) \in \ker \mathbb{M}_2(\zeta)$. Then from the expression of $\mathbb{M}_2(\zeta)$ (see (67)) we immediately deduce that $Y_k = 0$ for all $k \in [\![1, \ell_2 + r_2 - 1]\!]$. Then the first line of $\mathbb{M}_2(\zeta)$ reduces to $(\mathbb{A}_2^{r_2}(\zeta))^{-1}\mathbb{A}_2^{-\ell_2}(\zeta)X_{\ell_2+r_2} = 0$ which implies that $X_{\ell_2+r_2} = 0$ thanks to Assumption 3.1.

Now let κ be an eigenvalue of $\mathbb{M}_2(\zeta)$ we have on one hand, because $\mathbb{M}_2(\zeta)$ and $\mathscr{A}(\kappa_1, \kappa_2)$ is a companion matrices:

$$\begin{aligned}
\det(\mathbb{M}_2(\zeta) - \kappa I) &= (-1)^{N(\ell_2+r_2)} \det\left(\sum_{\mu_2=-\ell_2}^{r_2-1} \kappa^{\mu_2+\ell_2}(\mathbb{A}_2^{r_2}(\zeta))^{-1}\mathbb{A}_2^{\mu_2}(\zeta) + \kappa^{\ell_2+r_2} I\right), \\
&= (-1)^{N(\ell_2+r_2)}\kappa^{\ell_2 N} \det(\mathbb{A}_2^{r_2}(\zeta)^{-1}) \det\left(\sum_{\mu_2=-\ell_2}^{r_2} \kappa^{\mu_2}\mathbb{A}_2^{\mu_2}(\zeta)\right)
\end{aligned}$$

and on the other hand:

$$\begin{aligned}
\det(\mathscr{A}(e^{i\xi_1}, \kappa) - zI) &= (-1)^{Ns} \det\left(\sum_{\sigma=0}^{s} z^{s-\sigma}\widehat{Q^\sigma}(e^{i\xi_1}, \kappa) - z^{s+1} I\right) \\
&= (-1)^{N(s+1)} z^{N(s+1)} \det\left(\sum_{\mu_2=-\ell_2}^{r_2} \kappa^{\mu_2}\mathbb{A}_2^{\mu_2}(\zeta)\right).
\end{aligned}$$

by definition of the matrices $\mathbb{A}_2^{\mu_2}(\zeta)$. So as we have already shown that $\kappa \neq 0$, $\det(\mathbb{M}_2(\zeta) - \kappa I)$ and $\det(\mathscr{A}(e^{i\xi_1}, \kappa) - zI)$ vanish for the same values of κ, $z \in \mathscr{U}$.

Then we justify that $|\kappa| \neq 1$. By contradiction let $\kappa \in \mathbb{S}^1$ be an eigenvalue of $\mathbb{M}_2(\zeta)$ for $z \in \mathscr{U}$. It implies that z is an eigenvalue of $\mathscr{A}(e^{i\xi_1}, \kappa)$. But from the stability assumption of the scheme we have the von Neumann condition so that $|z| \leq 1$ is incompatible with the fact that z lies in $\mathscr{U}$.

The fact that there are exactly $N\ell_2$ eigenvalues in $\mathbb{D}$ comes from a connectedness argument and the application of Rouché's theorem. It follows exactly the arguments given in [Cou13]. So it will not be reproduced here.

□

We are now able to reformulate the uniform Kreiss-Lopatinskii condition under the following form:

Corollary 6.1 (Reformulation of the uniform Kreiss-Lopatinskii condition) *Under Assumption 3.1, also assume that the discretization for the Cauchy problem is strongly stable in the sense of Definition 3.1. If the scheme* (32) *is strongly stable in the sense of Definition 3.2 then for all $R \geq 2$ there exists C_R such that for all $\zeta \in \mathscr{U} \times \mathbb{R}$ satisfying $|z| \geq R$ the restriction of $\mathbb{B}_2(\zeta)$ to the stable subspace $E_2^s(\zeta)$ is an isomorphism. We denote its inverse by $\phi_2(\zeta)$.*

So far we have show the fact that the uniform Kreiss-Lopatinskii condition on the side $\partial\Omega_2$ is a necessary condition for (32) to be well-posed. We now turn to the description of the new necessary condition for strong stability. This condition involves two trace operators that to the value of the trace of the scheme on a side of the boundary makes correspond the value of the trace on the other boundary and have to be seen as discretized versions of the operators $\mathbb{T}_1$ and $\mathbb{T}_2$ in the corner condition described in Section 2.

Following the proof given in the continous setting (see Section 2) we use the decomposition:

$$\mathbb{C}^{N(\ell_2+r_2)} = E_2^s(\zeta) \oplus E_2^u(\zeta), \tag{72}$$

to decompose $\mathscr{X}_{j_2}$ the solution (69) in a stable and an unstable components:

$$\mathscr{X}_{j_2} := \Pi_2^s(\zeta)\mathscr{X}_{j_2} + \Pi_2^u(\zeta)\mathscr{X}_{j_2},$$

where $\Pi_2^s(\zeta)$ (resp.$\Pi_2^u(\zeta)$) denotes the projection upon $E_2^s(\zeta)$ (resp. $E_2^u(\zeta)$) with respect to the decomposition (72). We then use discrete Duhamel's formula[1] to obtain that each of these components is given by:

$$\Pi_2^s(\zeta)\mathscr{X}_{j_2} := \mathbb{M}_2(\zeta)^{j_2-1}\Pi_2^s(\zeta)\mathscr{X}_1 + \sum_{k=1}^{j_2-1} \mathbb{M}_2(\zeta)^{j_2-1-k}\Pi_2^s(\zeta)\mathscr{F}_{2,k}(V), \tag{73}$$

$$\Pi_2^u(\zeta)\mathscr{X}_{j_2} := -\sum_{k=j_2}^{+\infty} \mathbb{M}_2(\zeta)^{j_2-1-k}\Pi_2^u(\zeta)\mathscr{F}_{2,k}(V). \tag{74}$$

we used the fact that $\mathbb{M}_2(\zeta)$ is invertible. Then from the uniform Kreiss-Lopatinskii condition we know that at the level of the boundary $\partial\Omega_2$ the stable component of the trace is a function of the unstable component. More precisely we have:

$$\Pi_2^s(\zeta)\mathscr{X}_1 := \phi_2(\zeta)\left(\mathscr{G}_2(V) - \mathbb{B}_2(\zeta)\Pi_2^u(\zeta)\mathscr{X}_1\right),$$

from which we deduce that the trace of $\mathscr{X}_{j_2}$ is given by:

$$\begin{aligned} \mathscr{X}_1 &= \Pi_2^s(\zeta)\mathscr{X}_1 + \Pi_2^u(\zeta)\mathscr{X}_1, \\ &= \phi_2(\zeta)\mathscr{G}_2(V) + [\phi_2(\zeta)\mathbb{B}_2(\zeta) - I]\sum_{k=1}^{+\infty} \mathbb{M}_2(\zeta)^{j_2-1-k}\Pi_2^u(\zeta)\mathscr{F}_{2,k}(V). \end{aligned} \tag{75}$$

Taking the reverse Fourier transform of $\mathscr{X}_1$ in terms of ξ_1 such gives the relation:

$$(\mathscr{F}^{-1}\mathscr{X}_1)(x_1) = (P_2^{dis}\mathscr{G}_2(V))(x_1) + (\mathbb{T}_{1\to 2}^{dis}\mathscr{F}_{2,j_2}(V))(x_1), \tag{76}$$

where the operators P_2^{dis} and $\mathbb{T}_{1\to 2}^{dis}$ are defined by: $\forall U \in \ell^2([\![1-\ell_2, +\infty[\![$

$$(\mathbb{T}_{1\to 2}^{dis}U_j)(x_1) = (\mathbb{T}_{1\to 2}^{dis}U_j)(z, x_1) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_1\xi_1}\,[\phi_2(\zeta)\mathbb{B}_2(\zeta) - I]\sum_{k=1}^{+\infty} \mathbb{M}_2(\zeta)^{j-1-k}\Pi_2^u(\zeta)U\, d\xi_1, \tag{77}$$

$$(P_2^{dis}U)(x_1) = (P_2^{dis}U)(z, x_1) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_1\xi_1}\phi_2(\zeta)U\, d\xi_1. \tag{78}$$

[1]Note that to use Duhamel's formula we used the fact that $\mathbb{M}_2(\zeta)$ is invertible which is suitable from Lemma 6.1.

From the definition of $\mathscr{X}_1$ we have,

$$(\mathscr{F}^{-1}\mathscr{X}_1)(x_1) = (\mathscr{F}^{-1}X_{r_2}, \cdots, \mathscr{F}^{-1}X_{1-\ell_2})(x_1) = (W_{r_2}, \cdots, W_{1-\ell_2})(x_1).$$

Consequently, by definition of the step function W_{j_2}, evaluate (76) for $x_1 = j_1$, $j_1 \geq 1 - \ell_1$ gives the values of W_j for $j \in \overline{\mathscr{B}_2}$. By construction, these values coincide with $(V_j)_{j\in\overline{\mathscr{B}_2}}$. We recall that under the restriction $b_{21} \leq \ell_1 + r_1$ the error term on the boundary $\mathscr{G}_2(V)$ only involves the values of (V_j) for $j \in \mathscr{B}_1$. The error term in the interior $\mathscr{F}_{2,j_2}(V)$ involves the values of the (V_j) for $j \in \overline{\mathscr{B}_1}$, independently of the choice of the parameters.

Consequently, (76) evaluated for $x_1 = j_1$, $j_1 \geq 1 - \ell_1$ gives the value of $(V_j)_{j\in\overline{\mathscr{B}_2}}$ in terms of $(V_j)_{j\in\overline{\mathscr{B}_1}}$ and this equation of compatibility as to be seen as the "discretized" version of (19) which is the compatibility condition in PDE framework. The only difference about these two compatibility conditions is that due to the discret nature of finite difference schemes, there is an extra error term on the boundary in (76) compared to (19). However, in the particular setting $b_{21} = 0$, it has been see that this term vanish and in that case (76) becomes:

$$(\mathscr{F}^{-1}\mathscr{X}_1)(x_1) = (\mathbb{T}^{dis}_{1\to2}\mathscr{F}_{j_2}(V))(x_1),$$

which is totally analogous to (19).

Note that $\mathscr{F}_{j_2}(V)$ contains the matrix $(\mathbb{A}_2^{r_2}(z,\xi_1))^{-1}$ and that $\mathcal{F}_{2,j_2}$ The operator $\mathbb{T}^{dis}_{1\to2}$ seems to be a "discretized" version of the operator $\mathbb{T}_{1\to2}$ given in equation (20) in the PDE framework.

We thus have shown that: $\forall j_1 \in [\![1, +\infty[\![$,

$$V_{|\overline{\mathscr{B}_2}} := (V_{j_1,r_2}, \ldots, V_{j_1,1-\ell_2}) = (P_2^{dis}\mathscr{G}_2(V_{|\overline{\mathscr{B}_1}}))(j_1) + (\mathbb{T}^{dis}_{1\to2}\mathscr{F}_{2,j_2}(V_{|\overline{\mathscr{B}_1}}))(j_1). \tag{79}$$

Finally we reiterate the analysis made in Sections 5-6 but with an extension by zero for $j_2 \leq 1 - \ell_2$ this time. For such an extension, because the source term $(g_{1,j})_j$ is not zero, the error terms in the interior and on the boundary are respectively given by $\mathcal{F}_{1,j_2}(V)$ and $\mathcal{G}_1(g_1, V)$ (see (57) and (58)). We use the formulation:

$$\tag{80}$$

$$\mathcal{G}_1(g_1, V) = L_2(V)_j \mathbf{1}_{[\![1-\ell_2-b_{12},-\ell_2]\!]}(j_2) + g_{1,j}\mathbf{1}_{[\![1-\ell_2,+\infty[\![}(j_2) := \mathcal{G}_1(V)_j + \mathcal{G}_1(g_1)_j. \tag{81}$$

This new extension leads us to a compatibility condition, analogous to (76), between the value of the trace on $\partial\Omega_1$ in terms of the trace on $\partial\Omega_2$. More precisely we obtain:

$$(\mathscr{F}^{-1}\mathscr{X}_2)(x_2) = (P_1^{dis}\mathscr{G}_1(g_1, V))(x_2) + (\mathbb{T}^{dis}_{2\to1}\mathscr{F}_{1,j_2}(V))(x_2), \tag{82}$$

where the operators P_1^{dis} and $\mathbb{T}^{dis}_{2\to1}$ have an analogous expression than the operators P_2^{dis} and $\mathbb{T}^{dis}_{1\to2}$. More precisely, they are defined by: $\forall U \in \ell^2([\![1-\ell_1, +\infty[\![)$,

$$(\mathbb{T}^{dis}_{2\to1}U_j)(x_2) = (\mathbb{T}^{dis}_{2\to1}U_j)(z, x_2) \quad := \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}\left[\phi_1(\zeta)\mathbb{B}_1(\zeta) - I\right]\sum_{k=1}^{+\infty}\mathbb{M}_1(\zeta)^{j-1-k}\Pi_1^u(\zeta)U\, d\xi_2, \tag{83}$$

$$(P_1^{dis}U)(x_2) = (P_1^{dis}U)(z, x_2) \quad := \quad \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}\phi_1(\zeta)U\, d\xi_2, \tag{84}$$

where the matrices are defined by:

$$\mathbb{M}_1(z,\xi_2) := \begin{bmatrix} -(\mathbb{A}_1^{r_1}(z,\xi_2))^{-1}\mathbb{A}_1^{r_1-1}(z,\xi_1) & \ldots & \ldots & -(\mathbb{A}_1^{r_1}(z,\xi_2))^{-1}\mathbb{A}_1^{-\ell_1}(z,\xi_2) \\ I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & I & 0 \end{bmatrix},$$

$$\mathbb{B}_1(z,\xi_2) := \begin{bmatrix} 0 & \cdots & 0 & -\mathbb{B}_1^{b_{11},0} & -\mathbb{B}_1^{0,0} & I & & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \ddots & \\ 0 & \cdots & 0 & -\mathbb{B}_1^{b_{11},1-\ell_1} & -\mathbb{B}_1^{0,1-\ell_1} & 0 & & I \end{bmatrix},$$

with,

$$\forall \mu_1 \in [\![0, b_{11}]\!], \forall j_1 \in [\![1-\ell_1, 0]\!],\ \mathbb{B}_1^{\mu_1, j_1}(z, \xi_2) := \sum_{\sigma=-1}^{s} \sum_{\mu_2=0}^{b_{12}} z^{-(\sigma+1)} e^{ij_2\xi_2} B_1^{\sigma,\mu,j_1}.$$

Finally, the terms $\mathscr{G}_1(V)$ and $\mathscr{F}_{1,j_2}(V)$ appearing in 82 are given by:

$$\begin{array}{rcl}
\mathscr{F}_{1,j_1}(V) = \mathscr{F}_{1,j_1}(\zeta, V) & := & (-(\mathbb{A}_1^{r_1}(z,\xi_2))^{-1}\widehat{\mathcal{F}_{j_1}(V)}, 0, \dots, 0) \\
\mathscr{G}_1(g_1, V) = \mathscr{G}_1(z, g_1, V) & := & (\mathcal{G}_1\widehat{(g_1, V)}_0, \dots, \mathcal{G}_1\widehat{(g_1, V)}_{1-\ell_1}) \\
& = & (\widehat{\mathcal{G}_1(g_1)}_0, \dots, \widehat{\mathcal{G}_1(g_1)}_{1-\ell_1}) + (\widehat{\mathcal{G}_1(V)}_0, \dots, \widehat{\mathcal{G}_1(V)}_{1-\ell_1}), \\
& := & \mathscr{G}_1(g_1) + \mathscr{G}_1(V)
\end{array}$$

where we used the decomposition (81).

By linearity of P_1^{dis}, the compatibility condition (82) equivalently reads:

$$(\mathscr{F}^{-1}\mathscr{X}_2)(x_2) = (P_1^{dis}\mathscr{G}_1(V))(x_2) + (\mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_2}(V))(x_2) + (P_1^{dis}\mathscr{G}_1(g_1))(x_2). \tag{85}$$

Under the restriction $b_{12} \leq \ell_2 + r_2$, the right hand side of (85) only involves values of V_j for $j \in \overline{\mathscr{B}_2}$. Reitering the same arguments as those described in the beginning of this paragraph we thus obtain, by definition of $(W_j)_j$, the compatibility condition: $\forall j_2 \geq 1$,

$$V_{|\overline{\mathscr{B}_1}} := (V_{r_1,j_2}, \dots, V_{1-\ell_1,j_2}) = (P_1^{dis}\mathscr{G}_1(V_{|\overline{\mathscr{B}_2}}))(j_2) + (\mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_2}(V_{|\overline{\mathscr{B}_2}}))(j_2) + (P_1^{dis}\mathscr{G}_1(g_1))(j_2). \tag{86}$$

Combining (79) and (86) thus gives:

$$\begin{array}{rcl}
V_{|\overline{\mathscr{B}_1}} & = & \left(P_1^{dis}\mathscr{G}_1((P_2^{dis}\mathscr{G}_2(V_{|\overline{\mathscr{B}_1}}) + \mathbb{T}_{1\to2}^{dis}\mathscr{F}_{2,j_2}(V_{|\overline{\mathscr{B}_1}}))(j_1))\right)(j_2) \\
& + & \left(\mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_1}((P_2^{dis}\mathscr{G}_2(V_{|\overline{\mathscr{B}_1}}) + \mathbb{T}_{1\to2}^{dis}\mathscr{F}_{2,j_2}(V_{|\overline{\mathscr{B}_1}}))(j_1))\right)(j_2) + (P_1^{dis}\mathscr{G}_1(g_1))(j_2),
\end{array} \tag{87}$$

That we rewrite under the form:

$$(I - \mathbb{T}^{dis}(z))V_{|\overline{\mathscr{B}_1}} = (P_1^{dis}\mathscr{G}_1(g_1)), \tag{88}$$

where, from linearity, for $z \in \mathscr{U}$, the operator $\mathbb{T}^{dis}(z) : \ell^2(\overline{\mathscr{B}_1}) \mapsto \ell^2(\overline{\mathscr{B}_1})$ is defined by: $\forall V \in \ell^2(\overline{\mathscr{B}_1})$

$$\begin{array}{rcl}
\mathbb{T}^{dis}(z)V & := & P_1^{dis}\mathscr{G}_1(P_2^{dis}\mathscr{G}_2(V)) + P_1^{dis}\mathscr{G}_1(\mathbb{T}_{1\to2}^{dis}\mathscr{F}_{2,j_2}(V)) + \mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_1}(P_2^{dis}\mathscr{G}_2(V)) \\
& + & \mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_1}(\mathbb{T}_{1\to2}^{dis}\mathscr{F}_{2,j_2}(V)),
\end{array} \tag{89}$$

recall that all the terms in the right hand side of (89) depend on $z \in \mathscr{U}$.

Remark Let us stress that in the particular framework, $b_{12} = 0$ (resp. $b_{21} = 0$) we have $\mathscr{G}_2(V) = 0$ (resp. $\mathscr{G}_1(V) = 0$) for all V. Consequently for $b_{12} = b_{21}$ the operator $\mathbb{T}^{dis}$ becomes:

$$\mathbb{T}^{dis}(z)V = \mathbb{T}_{2\to1}^{dis}\mathscr{F}_{1,j_1}(\mathbb{T}_{1\to2}^{dis}\mathscr{F}_{2,j_2}(V)),$$

and consequently for such parameters there is no error on the boundaries and (88) as a form which is really close of the form of Osher's corner condition (23).

Thus we have shown the following result, which is the main result of this paper:

Theorem 6.1 *Under Assumption 3.1, also assume that the discretization for the Cauchy problem is strongly stable in the sense of Definition 3.1. Assume that the parameters b_{12} and b_{21} appearing in (33) satisfy the restrictions:*

$$b_{12} \leq \ell_2 + r_2 \text{ and } b_{21} \leq \ell_1 + r_1.$$

If the scheme (32) is strongly stable in the sense of Definition 3.2 then for all $z \in \mathscr{U}$ the restriction of V the solution of (37) to $\overline{\mathscr{B}_1}$ satisfies the compatibility condition (88).

Formally, if we believe that the source term on the side on the boundary $\mathscr{B}_1$ determines the value of the trace of the solution of the scheme then it seems natural to impose, as in the continous setting, that the operator $(I - \mathbb{T}^{dis}(z))$ is invertible on $\ell^2(\overline{\mathscr{B}_1})$. Moreover, as we are looking for energy estimates (38), where the constant C does not depend on $z \in \mathscr{U}$, we also ask that this inversibility property is uniform in terms of $z \in \mathscr{U}$. Thus we introduce the following definition:

Definition 6.1 (Discrete Osher's condition) *Under Assumption 3.1, also assume that the discretization for the Cauchy problem is strongly stable in the sense of Definition 3.1. We say that the finite difference scheme approximation* (32) *satisfies the discrete Osher's corner condition if the operator $(I - \mathbb{T}^{dis}(z))$ is uniformly invertible on $\ell^2(\overline{\mathscr{B}_1})$, this means that there exists $C > 0$ such that:*

$$\forall z \in \mathscr{U},\, \forall V \in \ell^2(\overline{\mathscr{B}_1}), \;\; \sum_{j \in \overline{\mathscr{B}_1}} |V_j|^2 \leq \sum_{j \in \overline{\mathscr{B}_1}} |((I - \mathbb{T}^{dis}(z))V)_j|^2.$$

7 Examples and numerical results.

7.1 Examples of explicit computations for the traces operators $\mathbb{T}^{dis}_{1\to 2}$ and $\mathbb{T}^{dis}_{2\to 1}$

7.2 Numerical results

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